Some Exact Sequences in the Higher K-theory of Rings

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§1. Introduction

A general K-theory exact sequence of localizations has been announced recently by Quillen [10]. For a Dedekind ring R with field of fractions F this sequence is

1.1) $$\cdots \to K_nR \to K_nF \to \coprod_{\mathfrak{p}\in \max R} K_{n-1}(R/\mathfrak{p}) \to K_{n-1}(R) \to \cdots \ .$$

There is some evidence now to suggest that there is a local-global principle valid in K-theory: the exact sequence 1.1) should be a consequence of conjectural short exact sequences

1.2) $$0 \to K_n(R_{\mathfrak{p}}) \to K_nF \to K_{n-1}(R/\mathfrak{p}) \to 0 \ .$$

The glue connecting the conjectural sequences 1.2) is the K-theory spectral sequence of Brown and Gersten [2], [7]

$$E_2^{pq} = H^p(\text{spec } R, \underline{\underline{K}}_{-q}) \Rightarrow K_{-p-q}(R).$$

(The glueing argument is given in Gersten [7].)

As evidence for 1.2) we remark that it is true if $n \le 2$ by a theorem of Dennis and Stein [5]. One of our objectives here is to prove

Theorem 1.3. If A is a discrete valuation ring with maximal ideal $\underline{m}$, then the transfer map $K_n(A/\underline{m}) \to K_n(A)$ in the localization sequence 1.1) is zero if either

a) $A/\underline{m}$ is finite, or

b) A is a k-algebra (k a field) where $A/\underline{m}$ is a finite separable extension of k.

The exactness of 1.1) together with 1.3) imply that 1.2) is exact under the hypotheses of 1.3).

The next result is included here because the techniques of proof are similar to those of 1.3,b).

Theorem 1.4. If F is a field, then these are short exact sequences

$$0 \to K_n(F) \to K_n(F(t)) \to \coprod_{\mathfrak{p} \in \max(F[t])} K_{n-1}(F[t]/\mathfrak{p}) \to 0 .$$

This is a consequence of the fact to be established that the transfer map $K_n(F[t]/\mathfrak{p}) \to K_n(F[t])$ in the localization sequence is zero.

We assume several results of Quillen [10], namely the localization sequence and the fact that if R is left regular (i.e., R is unitary, left noetherian, and each finitely generated left R module has finite projective dimension) then the map $K_n(R) \to K_n(R[t])$ of Quillen K-groups is an isomorphism. The results in this paper could be stated independently of Quillen [10], which has not yet appeared, in the form that certain transfer maps vanish. But to do so would distort the context in which these results have significance, so we have chosen what we believe is the lesser evil.

The paper is terminated by a somewhat oversized section on the theme of local-global principles in algebraic K-theory. That section is mostly conjectural and the content can be succinctly summarized by the suggestion that the sheaves $\underline{K}_n$ on a regular scheme X should be Cohen-Macauley (with respect to the filtration of codimension of supports) in the terminology of Grothendieck [9, p. 238]. This conjecture unifies a number of phenomena of local algebra and includes some questions raised by Claborn and Fossum [3]. It also is intimately related to the cycle map into Hodge cohomology, as we shall show. More important, in our opinion, it indicates a direction of research that should be pursued in applying the higher K-functors to questions about cycles in algebraic geometry.

We want to thank Spencer Bloch for patiently listening to these conjectures in their formative stage, commenting on them, and indicating to us the analogous questions that exist in deRham cohomology. Without his encouragement, this article could not have been written.

§2. Producing maps of Quillen K-groups.

We need a universal property of Quillen's space $B^+_{G\ell A}$ to describe and compute the transfer map in the localization sequence 1.1). Let A be a unitary ring and let $\underline{P}_A$ be the category of finitely generated projective left A-modules. If G is a group, then $\underline{P}_A(G)$ denotes the category of G-representations in $\underline{P}_A$, and $R_A(G)$ denotes the Grothendieck group of $\underline{P}_A(G)$ with relations all short exact sequences of representations. We recall a result of Quillen (see [6], Theorem 2.6).

Theorem 2.1. Let X be a finite based CW complex. There is a natural transformation (natural in X)

$$R_A(\pi_1 X) \xrightarrow[X]{\eta} [X, K_0(A) \times B^{+}_{G\ell(A)}]$$

which is universal for morphisms

$$R_A(\pi_1 X) \xrightarrow[X]{\xi} [X, H] \ ,$$

where H is an H-space. That is, there are maps

$$K_0(A) \times B^{+}_{G\ell(A)} \xrightarrow{\omega} H \ ,$$

such that the diagram

$$\begin{array}{ccc} R_A(\pi_1 X) & \longrightarrow & [X, H] \\ \downarrow{\scriptstyle \eta_X} & \nearrow{\scriptstyle \omega_*} & \\ [X, K_0(A) \times B^{+}_{G\ell(A)}] & & \end{array}$$

commutes, and the morphism of functors

$$[\cdot, K_0(A) \times B^{+}_{G\ell(A)}] \xrightarrow{\omega_*} [\cdot, H]$$

is independent of the choice of ω.

As an application of this result, suppose that one has a natural transformation $R_A(G) \xrightarrow{f_G} R_B(G)$ of (contravariant) functors of G. Then from the diagram

$$\begin{array}{ccc} R_A(\pi_1 X) & \longrightarrow & R_B(\pi_1 X) \\ \downarrow & & \downarrow \\ [X, K_0(A) \times B^{+}_{G\ell(A)}] & \xrightarrow{\omega} & [X, K_0(B) \times B^{+}_{G\ell B}] \end{array}$$

one sees that, taking $X = S^n$, there is induced a map $K_n(A) \to K_n(B)$.

§3. The transfer for Dedekind rings.

If R is a noetherian ring, let $\underline{M}_R$ denote the category of finitely generated R modules. If R is regular, then the inclusion functor $\underline{P}_R \hookrightarrow \underline{M}_R$ induces an isomorphism of K-groups by Quillen [10]. If R is in addition Dedekind and $\mathfrak{p}$ is a maximal ideal of R, then the transfer map $K_n(R/\mathfrak{p}) \to K_n(R)$ is induced by the functor $\underline{P}_{R/\mathfrak{p}} \hookrightarrow \underline{M}_R$, given by restricting operators from $R/\mathfrak{p}$ to R via the canonical map $R \to R/\mathfrak{p}$. We shall compute the transfer by using representation of groups in projective modules.

Theorem 3.1. Suppose that R is a Dedekind ring and that either

a) $R/\mathfrak{p}$ is finite for each maximal ideal $\mathfrak{p}$, or

b) R is a k-algebra, where k is a field, and $R/\mathfrak{p}$ is finite over k for each maximal ideal $\mathfrak{p}$ of R. Then

1.) Each V in $\underline{P}_{R/\mathfrak{p}}(G)$ has a resolution over R[G]

$$0 \to P_1 \to P_0 \to V \to 0$$

with P_i in $\underline{P}_R(G)$,

2.) The natural map $K_0(\underline{P}_{R/\mathfrak{p}}(G)) \to K_0(\underline{M}_R(G))$ lifts through the cartan map

$$c\colon K_0(\underline{P}_R(G)) \to K_0(\underline{M}_R(G)) ,$$

where c is induced by the inclusion functor

$$\underline{P}_R(G) \to \underline{M}_R(G) ;$$

that is, there is a commutative diagram

$$\begin{array}{ccc} & & K_0(\underline{P}_R(G)) \\ & \overset{\rho_{\mathfrak{p}}}{\nearrow} & \Big\downarrow c \\ K_0(\underline{P}_{R/\mathfrak{p}}(G)) & \longrightarrow & K_0(\underline{M}_R(G)) \end{array}$$

with $\rho_{\mathfrak{p}}$ natural in G; and

3.) The cartan map

$$c\colon K_0(\underline{P}_R(G)) \to K_0(\underline{M}_R(G))$$

is an isomorphism for all groups G.

<u>Proof</u>. Suppose V is in $\underline{P}_{R/\mathfrak{p}}(G)$ and $R/\mathfrak{p}$ is finite. Then there is a factorization

$$\begin{array}{ccc} G & \xrightarrow{\ \varphi\ } & \mathrm{Aut}_{R/\mathfrak{p}}(V) \\ & \searrow \quad \nearrow & \\ & G_1 & \end{array}$$

where φ is the structure map of V and G_1 is finite. Hence V can be considered an $R[G_1]$ module, and a standard argument (Swan [14], Theorem 1.2), shows that there is a resolution

$$0 \to P_1 \to P_0 \to V \to 0$$

with P_i projective and finitely generated as $R[G_1]$ modules. If we restrict operators from G_1 to G, this yields a resolution with P_i in $\underline{P}_R(G)$.

Suppose now that R is a Dedekind k-algebra, k a field, and $R/\mathfrak{p}$ is finite over k. Let V be in $\underline{P}_{R/\mathfrak{p}}(G)$. If $\underline{V}$ is the underlying k-vector space with G action, $\underline{V}$ is in $\underline{P}_k(G)$. Hence $R \underset{k}{\otimes} \underline{V}$ is in $\underline{P}_R(G)$ where the action of $g \in G$ is given by

$$a(r\otimes v) = r \otimes gv; \; r \in R, \; v \in \underline{V}.$$

Define a map $w\colon R \underset{k}{\otimes} \underline{V} \to V$ by $w(r\otimes v) = \bar{r}v$, where $\bar{r}$ is the class of r in $R/\mathfrak{p}$. Clearly w is G-linear and

$$0 \longrightarrow \operatorname{Ker} w \longrightarrow R \underset{k}{\otimes} \underline{V} \xrightarrow{w} V \longrightarrow 0$$

is the desired resolution with Ker w and $R \underset{k}{\otimes} \underline{V}$ in $\underline{P}_R(G)$. This completes the proof of conclusion 1) in both cases a) and b).

The proof of 2) follows from 1) in a standard way. One defines an association

$$\text{objects } \underline{P}_{R/\mathfrak{p}}(G) \longrightarrow K_0(\underline{P}_R(G))$$

by $[V] \longmapsto [P_0] - [P_1]$ where

$$0 \to P_1 \to P_0 \to V \to 0$$

is an exact sequence of $R[G]$ modules, V in $\underline{P}_{R/\mathfrak{p}}(G)$ and P_0, P_1 in $\underline{P}_R(G)$. A Schanuel lemma type argument shows the association is independent of the resolution chosen and additive over short exact sequences. Thus the map $\rho_{\mathfrak{p}}\colon K_0(\underline{P}_{R/\mathfrak{p}}(G)) \to K_0(\underline{P}_R(G))$ is defined and the diagram

$$\begin{array}{ccc} & & K_0(\underline{P}_R(G)) \\ & \overset{\rho_{\mathfrak{p}}}{\nearrow} & \big\downarrow c \\ K_0(\underline{P}_{R/\mathfrak{p}}(G)) & \longrightarrow & K_0(\underline{M}_R(G)) \end{array}$$

commutes.

We begin now the proof of 3). Let $\underline{T}_R$ be the category of finitely generated torsion R modules and let $\underline{T}_R(G)$ be the category of G representations in $\underline{T}_R$. A simple devissage argument establishes

<u>Lemma 3.2</u>. $$K_0(\underline{T}_R(G)) \cong \coprod_{\substack{\mathfrak{p} \text{ maximal} \\ \text{in } R}} K_0(\underline{P}_{R/\mathfrak{p}}(G)) .$$

Hence, as a result of conclusion 2) one deduces

<u>Lemma 3.3</u>. Under the hypotheses of 3.1 the map $K_0(\underline{T}_R(G)) \to K_0(\underline{M}_R(G))$ induced by inclusion $\underline{T}_R(G) \hookrightarrow \underline{M}_R(G)$ lifts through the cartan map. That is, there is a commutative diagram

$$\begin{array}{ccc} & & K_0(\underline{P}_R(G)) \\ & \overset{\varphi}{\nearrow} & \Big\downarrow c \\ K_0(\underline{T}_R(G)) & \longrightarrow & K_0(\underline{M}_R(G)) \end{array}$$

We establish now the surjectivity of c. If M is in $\underline{M}_R(G)$, then $[M] = [M_{tor}] + [M/M_{tor}]$ in $K_0(\underline{M}_R(G))$ where M_{tor} is the torsion submodule of M, with G action. But M/M_{tor} is in $P_R(G)$ and $[M_{tor}]$ is in the image of c by Lemma 3.3. Hence $[M]$ is in the image of c.

Since c is surjective, to prove it is injective it suffices to exhibit a left inverse ψ: $K_0(\underline{M}_R(G)) \to K_0(\underline{P}_R(G))$ for c. If M is in $\underline{M}_R(G)$, then we send M to $[M/M_{tor}] + \varphi[M_{tor}]$ in $K_0(\underline{P}_R(G))$, where φ is defined in 3.3. The problem is to prove that this association is additive over short exact sequences.

Lemma 3.4. Given a short exact sequence in $\underline{M}_R(G)$,

$$0 \to X \to Y \to Z \to 0 ,$$

where X and Y are in $\underline{P}_R(G)$ and Z is in $\underline{T}_R(G)$, then

$$[Y] = [X] + \varphi[Z]$$

in $K_0(\underline{P}_A(G))$.

Proof. By a devissage argument, we may assume that Z is an $R/\mathfrak{p}$ module for some maximal ideal $\mathfrak{p}$. In that case, $\varphi(Z) = [Y] - [Z]$ by definition (compare 3.2 and 3.3).

Lemma 3.5. The association

$$\text{objects } \underline{M}_R(G) \xrightarrow{\psi} K_0(\underline{P}_R(G))$$

given by $M \longmapsto [M/M_{tor}] + \varphi[M_{tor}]$ is additive over short exact sequences.

Proof. Consider the short exact sequence in $\underline{M}_R(G)$

$$0 \to M' \to M \to M'' \to 0 .$$

Observe that the sequence

$$0 \to M'_{tor} \to M_{tor} \to M''_{tor}$$

is exact. We embed these sequences in the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & M'_{tor} & \to & M' & \to & M'/M'_{tor} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow i & & \\
0 & \to & M_{tor} & \to & M & \to & M/M_{tor} & \to & 0 \\
 & & \downarrow u & & \downarrow & & \downarrow j & & \\
0 & \to & M''_{tor} & \to & M'' & \to & M''/M''_{tor} & \to & 0 \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

Consider the columns as chain complexes, look at the long exact homology sequence, and apply exactness in the first two columns. We denote by H( ) the homology of a column at ( ). We get $H(M'/M'_{tor}) = 0$,

$$H(M/M_{tor}) = \text{coker } u \ , \ H(M''/M''_{tor}) = 0 \ .$$

Thus we have exact sequences

$$0 \to M'/M'_{tor} \to \text{Ker } j \to \text{coker } u \to 0 \ ,$$

$$0 \to \text{Ker } j \to M/M_{tor} \to M''/M''_{tor} \to 0 \ .$$

By lemma 3.4, (since M'/M'_{tor} and Ker j are torsion free whereas coker u is torsion), we have in $K_0(\underline{P}_R(G))$

3.6) $[M'/M'_{tor}] + \varphi[\text{coker } u] = [\text{Ker } j] = [M/M_{tor}] - [M''/M''_{tor}]$.

But we have the exact sequence in $\underline{T}_R(G)$

$$0 \to M'_{tor} \to M_{tor} \to M''_{tor} \to \text{coker } u \to 0 \ ,$$

whence $\varphi(\text{coker } u) = \varphi[M'_{tor}] + \varphi[M''_{tor}] - \varphi[M_{tor}]$. Substituting into (3.6) and recombining shows that

$$\psi(M) = \psi(M') + \psi(M'') ,$$

whence ψ is additive. This completes the proof of lemma 3.5.

It follows that ψ factors to give a morphism (also denoted ψ)

$$K_0(\underline{M}_R(G)) \xrightarrow{\psi} K_0(\underline{P}_R(G)).$$

Clearly we have $\psi \circ c = 1$, and this completes the proof of Theorem 3.1

Remark. A. Dress has constructed an example of a discrete valuation ring A, residue class field k the algebraic closure of a finite field, such that the one dimensional standard representation of k^* over k cannot be resolved as in conclusion 1) of Theorem 3.1 (oral communication).

Suppose now that R is dedekind and either hypotheses a) or b) of Theorem 3.1 are valid for R. Then by conclusion 2) of 3.1 the maps $K_0(\underline{P}_{R/\mathfrak{p}}(G)) \to K_0(\underline{M}_R(G))$ lift through the Cartan map to maps $K_0(\underline{P}_{R/\mathfrak{p}}(G)) \xrightarrow{\rho_\mathfrak{p}} K_0(\underline{P}_R(G))$, natural in G. The argument following Theorem 2.1 shows that these morphisms induce maps $K_n(R/\rho) \to K_n(R)$.

Theorem 3.7. Let R be a dedekind ring and assume either a) or b) of 3.1 are satisfied by R. Let S be a set of integral primes (possibly empty) and let $\mathbb{Z}_S$ denote the localization of $\mathbb{Z}$ away from S, $\mathbb{Z}_S = \mathbb{Z}[p^{-1}, p \in S]$. Suppose that, for every group G, the natural map $R_{R/\mathfrak{p}}(G) \xrightarrow{\rho_\mathfrak{p}} R_R(G)$ just constructed is such that

$\rho_{\mathfrak{p}} \otimes \mathbf{Z}_S$ is zero. Then the transfer map tensored with $\mathbf{Z}_S$,

$$K_n(R/\mathfrak{p}) \otimes \mathbf{Z}_S \to K_n(R) \otimes \mathbf{Z}_S ,$$

is zero.

Proof. We have indicated the universal property satisfied by the space $K_0(A) \times B^+_{G\ell(A)}$ in Theorem 2.1. The localization of this space away from S (Sullivan [13]) satisfies a corresponding universal property, with $R_A(G) \otimes \mathbf{Z}_S$ replacing $R_A(G)$, for H spaces whose homotopy is a $\mathbf{Z}_S$ module. The result follows from the universal property.

§4. Proof of Theorem 1.3 in the case of finite residue class field.

Theorem 4.1. Let A be a discrete valuation ring with maximal ideal $\underline{m}$ and $A/\underline{m}$ finite of characteristic p. Then for every finite group G, the map $R_{A/\underline{m}}(G) \xrightarrow{\rho_m} R_A(G)$ of 3.1, conclusion 2, has p-torsion image in $R_A(G)$.

Proof. It suffices to prove that the map $R_{A/\underline{m}} \otimes \mathbf{Z}_{p^{-1}} \to R_A(G) \otimes \mathbf{Z}_{p^{-1}}$ is zero. Let $k = A/\underline{m}$. We shall construct a commutative diagram

$$\begin{array}{ccc} K_0(k[G]) & \dashrightarrow & K_0(A[G]) \\ \downarrow c & & \downarrow \\ R_k(G) = G_0(k[G]) & \longrightarrow & R_A(G) . \end{array}$$

The first vertical arrow is the usual Cartan map induced by

$\underline{P}_{k[G]} \hookrightarrow \underline{M}_{k[G]}$. The second vertical arrow is induced by $\underline{P}_{A[G]} \hookrightarrow \underline{P}_A(G)$. To construct the top horizontal arrow observe that if $0 \neq P \in$ object $\underline{P}_{k[G]}$, then by Kaplansky's theorem (Bass [1] p. 632), we have $hd_{A[G]}P = 1$. Thus restriction of operators induces

$$\underline{P}_{k[G]} \hookrightarrow \underline{H}_{A[G]}$$

Now the Cartan map c is known to be a monomorphism with cokernel a finite p-group (Serre [12], p. III - 13,§3.1). Thus

$$c \otimes 1\colon K_0(k[G]) \otimes \mathbb{Z}_{p^{-1}} \longrightarrow G_0(k[G]) \otimes \mathbb{Z}_{p^{-1}}$$

is an isomorphism. To complete the proof of 4.1, it thus suffices to prove

Lemma 4.2. The map $K_0(k[G]) \to K_0(A[G])$ is zero.

Proof. If $0 \neq V \in$ object $\underline{P}_{k[G]}$, then, as we remarked before, $hd_{A[G]}V = 1$, so there is a resolution

$$0 \to P_1 \to P_0 \to V \to 0$$

with $P_i \in$ object $\underline{P}_{A[G]}$. If F is the field of fractions of A, it follows that $F \otimes_A P_1 \cong F \otimes_A P_0$ as F[G] modules. By a theorem of Swan [14] (Theorem 1.10) it follows that $P_1 \cong P_0$ over A[G]. But the map $K_0(k[G]) \to K_0(A[G])$ is induced by $[V] \mapsto [P_0] - [P_1] = 0$. This completes the proof of 4.2 and Theorem 4.1 is fully established.

<u>Proof of 1.3 when $A/\underline{m}$ is finite of characteristic p.</u>

We show first that for every group G, the map

$$R_{A/m}(G) \to R_A(G)$$

of 3.1, conclusion 2, has p-torsion image. However, given any representation $G \xrightarrow{\varphi} \mathrm{Aut}_{A/\underline{m}}(V)$ in $\underline{P}_{A/\underline{m}}(G)$, one knows that φ factors through a finite quotient G_1 of G. By considering the diagram

$$\begin{array}{ccc} R_{A/\underline{m}}(G_1) & \longrightarrow & R_A(G_1) \\ \downarrow & & \downarrow \\ R_{A/\underline{m}}(G) & \longrightarrow & R_A(G) \end{array}$$

and applying Theorem 4.1, one sees that the image of [V] in $R_A(G)$ is p-torsion. Thus every generator of $R_{A/\underline{m}}(G)$ has p-torsion. Thus every generator of $R_{A/\underline{m}}(G)$ has p-torsion image in $R_A(G)$.

Hence the induced map

$$R_{A/\underline{m}}(G) \otimes \mathbb{Z}_{p^{-1}} \longrightarrow R_A(G) \otimes \mathbb{Z}_{p^{-1}}$$

is zero for all groups G. By Theorem 3.7, this implies that the transfer $K_n(A/\underline{m}) \to K_n(A)$, tensored with with $\mathbb{Z}_{p^{-1}}$, becomes zero. However by a theorem of Quillen [11], $K_i(A/\underline{m})$ is finite (for $i > 0$) without p-torsion. It follows that $K_i(A/\underline{m}) \to K_i(A)$ is zero for $i > 0$. For $i = 0$, the assertion is obvious. The proof of Theorem 1.3, case a), is complete.

An amusing consequence of Theorem 1.3, case a) is the following

Theorem 4.3. For each prime number p, the natural map

$$K_{2n+1}(\mathbb{Z}_{(p)}) \to K_{2n+1}(\mathbb{Q})$$

is an isomorphism for $n > 0$. In addition $K_{4n+3}(\mathbb{Z}_{(p)})$ is finite of order independent of the prime p.

Proof. The first statement follows from the localization Theorem 1.1, Theorem 1.3, case a), and Quillen's result [11] that $K_{2n}(\mathbb{F}_p) = 0$ for $n > 0$.

As for the second statement, consider the localization theorem for $\mathbb{Z}$:

$$\coprod_p K_{4n+3}(\mathbb{F}_p) \to K_{4n+3}\mathbb{Z} \to K_{4n+3}(\mathbb{Q}) \to 0 .$$

By Borel's theorem, $K_{4n+3}(\mathbb{Z})$ is torsion and by Quillen's theorem $K_{4n+3}(\mathbb{Z})$ is finitely generated. Thus $K_{4n+3}(\mathbb{Z})$ is finite as well as $K_{4n+3}(\mathbb{Q})$.

Using 5.5 we can settle negatively a question which was considered by several people at the Conference, whether $K_i(A) \to K_i(A/\underline{m})$ is surjective if A is a discrete valuation ring. For if we take $i = 4n+3$ and $A = \mathbb{Z}_{(p)}$, then $K_{4n+3}(\mathbb{F}_p)$ is cyclic of order $p^{2n+2} - 1$ by Quillen's computation [11]. However $K_{4n+3}(\mathbb{Z}_{(p)})$ is finite of some order independent of p. Thus for sufficiently large primes p, $K_{4n+3}(\mathbb{Z}_{(p)}) \to K_{4n+3}(\mathbb{F}_p)$ is not surjective.

Theorem 4.3 has some connection with one of a number of conjectures discussed by Lichtenbaum at the conference. We recall that Lichtenbaum has conjectured that $K_{4k-1}(R)$ is of order $w_{2k}(\mathbb{Q})$ for any subring R of $\mathbb{Q}$, where $w_{2k}(\mathbb{Q}) = 2$ denominator $(B_{2k}/2k)$.

Theorem 4.3 affirms that $K_{4n+3}(R)$ is of order independent of R for all local subrings of $\mathbb{Q}$.

§5. Proof of Theorem 1.3 in the equicharacteristic case.

Throughout this section the discrete valuation ring A is a k algebra (k a field) and the residue class field $A/\underline{m} = L$ is a finite separable extension of k.

Lemma 5.1. The kernel J of the natural map $L \underset{k}{\otimes} L \to L$ is generated by an idempotent e of the ring $L \underset{k}{\otimes} L$.

Proof. Since L is a separable extension of k, $L \underset{k}{\otimes} L$ is reduced. Since L is finite over k, $L \underset{k}{\otimes} L$ is semisimple, and consequently any ideal of $L \underset{k}{\otimes} L$ has an idempotent generator.

Lemma 5.2. $L[[t]] \underset{k}{\otimes} L \cong L \underset{k}{\otimes} L[[t]]$.

This is immediate since $\dim_k L < \infty$.

We define now a ring homomorphism $A \underset{k}{\otimes} L \to L$ by $a \otimes \ell \mapsto \bar{a}\cdot\ell$, where $\bar{a}$ is the class of a in $L = A/\underline{m}$.

Proposition 5.3. In the short exact sequence

$$0 \to N \to A \underset{k}{\otimes} L \to L \to 0,$$

N is a free $A \underset{k}{\otimes} L$ module of rank 1.

Proof. Choose $\pi \in \underline{m}$ with $A\pi = \underline{m}$. Then by the structure theorem for complete local rings, the completion $\hat{A}$ of A is isomorphic to $L[[\pi]]$, and

$$\hat{A} \underset{k}{\otimes} L \cong L \underset{k}{\otimes} L[[\pi]] \quad \text{(by 5.2)} .$$

By approximation, we may choose an element $x \in A \underset{k}{\otimes} L$, whose image in $\hat{A} \underset{k}{\otimes} L \simeq L \underset{k}{\otimes} L\,[[\pi]]$ is

$$e + \pi + \pi^2 d\ , \quad d \in L \underset{k}{\otimes} L\,[[\pi]]\ ,$$

where e is as in lemma 5.1. We claim that the sequence

$$0 \longrightarrow A \underset{k}{\otimes} L \xrightarrow{x} A \underset{k}{\otimes} L \longrightarrow L \longrightarrow 0$$

is exact. Since $\hat{A}$ is a faithfully flat A module, it suffices to prove that the sequence

$$0 \longrightarrow L \underset{k}{\otimes} L\,[[\pi]] \xrightarrow{e+\pi+\pi^2 d} L \underset{k}{\otimes} L\,[[\pi]] \longrightarrow L \longrightarrow 0$$

is exact. But this is an elementary computation that we omit. This completes the proof of 5.3.

Theorem 5.4. The map

$$R_L(G) \longrightarrow R_A(G)$$

of 3.1, conclusion 2, is zero for every group G.

Proof. Suppose $V \in \underline{P}_L(G)$, so we are given a homomorphism

$$\varphi\colon G \longrightarrow \mathrm{Aut}_L(V)\ .$$

We have just produced an exact sequence of A - L bimodules

$$0 \longrightarrow A \underset{k}{\otimes} L \longrightarrow A \underset{k}{\otimes} L \longrightarrow L \longrightarrow 0\ .$$

Hence, after $\underset{L}{\otimes} V$, we get an exact sequence of A-modules

$$0 \longrightarrow A \underset{k}{\otimes} V \longrightarrow A \underset{k}{\otimes} V \longrightarrow V \longrightarrow 0\ .$$

Observe that $A \otimes_k L$ is in $\underline{P}_A$ since $\dim_k L < \infty$. The G-structure on V makes this into an exact sequence of A[G] modules, and $A \otimes_k V$ is in $\underline{P}_A(G)$. Hence the map $R_L(G) \to R_A(G)$ is determined by $(V) \longmapsto (A \otimes_k V) - (A \otimes_k V) = 0$, and hence this map is zero. This completes the proof of 5.4

Theorem 1.3, case b, now follows immediately from 5.4 and 3.7, where one takes S to be empty. Thus the proof of Theoeem 1.3 is complete in all cases.

§6. Localization sequence of a polynomial ring.

Let F be a field and $\mathfrak{p}$ a prime ideal of the polynomial ring F[t]. Let $F(\mathfrak{p})$ denote $F[t]/\mathfrak{p}$.

Lemma 6.1. The map $\rho_{\mathfrak{p}}$ of Theorem 3.1, conclusion 2,

$$\rho_{\mathfrak{p}}: R_{F(\mathfrak{p})}(G) \longrightarrow R_{F[t]}(G) ,$$

is zero for any group G.

Proof. Suppose $G \xrightarrow{\psi} \mathrm{Aut}_{F(\mathfrak{p})}(V)$ is an object of $\underline{P}_{F(\mathfrak{p})}(G)$. Now $F(\mathfrak{p})$ is a monogenic field extension of F,

$$F(\mathfrak{p}) = F(\alpha), \text{ where } f(\alpha) = 0, \ f \in \mathfrak{p} .$$

Let τ be the element of $\mathrm{End}_F(V)$ represented by multiplication by α. Then

$$\tau \circ \psi(g) = \psi(g) \circ \tau \ , \ g \in G . \tag{6.2}$$

Examine the characteristic sequence of the endomorphism τ of V ([1], p. 630):

$$6.3) \qquad 0 \longrightarrow V[t] \xrightarrow{t-\tau} V[t] \xrightarrow{\varphi_\tau} V \longrightarrow 0$$

where $\varphi_\tau(\Sigma v_i t^i) = \Sigma\, \tau^i(v_i)$ and $(t-\tau)(\Sigma\, v_i t^i) = \Sigma(v_{i-1}-\tau(v_i))t^i$. Clearly $V[t]$ is finitely generated and free as $F[t]$ module. Also, $V[t]$ becomes a G module in the obvious way: $g(\Sigma\, v_i t^i) = \Sigma\, \psi(g)(v_i)\cdot t^i$, and 6.2) guarantees that the sequence 6.3) is exact as $F[t][G]$ modules. Hence the map $R_{F(\mathfrak{p})}(G) \longrightarrow R_{F[t]}(G)$ is induced by $V \longrightarrow [V[t]] - [V[t]] = 0$. This completes the proof of 5.1.

From Theorem 3.7 it follows that the transfer map $K_n(F(\mathfrak{p})) \longrightarrow K_n(F[t])$ is zero. But Quillen's theorem [11] states that $K_n(F) = K_n(F[t])$. If we combine these facts with Quillen's localization sequence 1.1), we deduce Theorem 1.4.

§7. Local-Global Principles.

Suppose that X is a separated noetherian regular scheme possessing an ample line bundle. Then there are two procedures for producing higher K-groups [7],[10] which lead to the same K theory for X, $K_n(X)$. We summarize here properties of this theory.

0) $K_0(X)$ is the Grothendieck group of the category of vector bundles on X.

1) If X = Spec A, then $K_n(X) = K_n(A)$.

2) If U and V are open subschemes of X, there is a Mayer-Vietoris sequence

$$\cdots \longrightarrow K_{n+1}(U\cap V) \longrightarrow K_n(U\cup V) \longrightarrow K_n(U)\oplus K_n(V) \longrightarrow K_n(U\cap V) \longrightarrow \cdots$$

3) There is a spectral sequence of cohomological type in the fourth quadrant, E_r^{pq}, with

$$E_1^{p,q} = \bigoplus_{\substack{\text{codim} D=p \\ D \text{ a prime} \\ \text{cycle}}} K_{-p-q}(F_D) \Rightarrow K_{-p-q}(X) ,$$

where F_D is the field of rational functions on D (Quillen [10]).

4) There is a spectral sequence of cohomological type in the fourth quadrant $'E_r^{p,q}$ with

$$'E_2^{p,q} = H^p(X,\underline{K}_{-q}) \Rightarrow K_{-p-q}(X) ,$$

where $\underline{K}_n$ is the sheaf associated to the presheaf $U \longmapsto K_n(\Gamma(U,\mathcal{O}_X))$ on the Zariski site (Brown and Gersten [2]).

We propose now

7.1) Working hypothesis (first form):

$$E_2^{p,q} = {'E_2^{p,q}}$$

Suppose now that A is a regular local ring. Then a simple computation shows that $H^n(\text{Spec } A, \underline{F}) = 0$ for any $n > 0$ and any abelian Zariski sheaf $\underline{F}$. Hence a first consequence of 7.1 is

7.2) Working hypothesis (second form): The differentials $d_1^{p,q}$ in the Quillen spectral sequence $E_1^{p,q}$ for a regular local ring A produce long exact sequences

$$0 \longrightarrow K_n(A) \longrightarrow K_n(F) \longrightarrow \bigoplus_{\substack{\text{ht } D=1 \\ D \text{ prime}}} K_{n-1}(F_D) \longrightarrow \bigoplus_{\substack{\text{ht } D=2 \\ D \text{ prime}}} K_{n-2}(F_D)$$

$$\longrightarrow \cdots \longrightarrow \bigoplus_{\substack{\text{ht } D=n-1 \\ D \text{ prime}}} F_D^* \longrightarrow Z^n \longrightarrow 0$$

where F is the field of fractions of A, F_D is the field of fractions of the domain A/D, and Z^n is the free abelian group with free basis the height n primes of A (Z^n is the group of codimension n cycles on Spec A).

If we globalize 7.2) (using the fact that exactness of sheaves is tested at points) we deduce immediately

7.3) Working hypothesis (third form):

If X is an irreducible noetherian regular scheme, then there is a long exact sequence of Zariski sheaves for each $n \geq 0$.

$$0 \longrightarrow \underline{K_n} \longrightarrow \underline{K_n(F)}_X \longrightarrow \bigoplus_{\substack{\text{codim } D=1 \\ D \text{ prime} \\ \text{cycle}}} \underline{K_{n-1}(F_D)}_D \longrightarrow \cdots$$

$$\longrightarrow \bigoplus_{\substack{\text{codim } D=n-1 \\ D \text{ prime cycle}}} \underline{F^*_D}_D \longrightarrow \underline{Z}^n \longrightarrow 0$$

where F_D is the field of rational functions on D, $\underline{K_i(F_D)}_D$ is the constant sheaf $K_i(F_D)$ on D extended by zero to all of X, and $\underline{Z}^n$ is the sheaf associated to the presheaf

$$U \longmapsto Z^n(U) ,$$

the codimension n cycle group of U.

Observe that each sheaf in 7.3) is flabby except for $\underline{K_n}$. Consequently, $H^p(X,\underline{K_n})$ can be calculated from this flabby resolution, by taking global sections. But the global sections are just the E_1^{**} terms of the spectral sequence $(E_r^{p,q})$, and the differentials correspond, so we have established

Proposition 7.4. The working hypothesis (third form) 7.3, implies the working hypothesis (first form) 7.1, so all statements, 7.1, 7.2, and 7.3 are equivalent.

Let us list some elementary consequences of the working hypothesis in the form of remarks.

Remark 1. If A is a discrete valuation ring with field of fractions F and maximal ideal $\underline{m}$, then 7.2) gives the short exact sequences

$$0 \longrightarrow K_n(A) \longrightarrow K_nF \xrightarrow{\partial} K_{n-1}(A/\underline{m}) \longrightarrow 0 ,$$

where the map ∂ is the tame symbol.

More generally the prediction is that for any regular local ring A, the map $K_nA \longrightarrow K_nF$ is injective, where F is the field of fractions of A.

Remark 2. Taking cohomology in 7.3) and using flabbyness of the sheaves, we deduce the exact sequences

7.5) $$\bigoplus_{\substack{\text{codim}D=i-1 \\ D \text{ irreducible reduced} \\ \text{closed subscheme}}} F_D^* \xrightarrow{\quad\partial\quad} Z^i \longrightarrow H^i(X,\underline{K_i}) \longrightarrow 0 .$$

The (conjectural) map $Z^i \longrightarrow H^i(X,\underline{K_i})$ has been proposed by S. Bloch [15], in the case i = 2, as the "second universal Chern class" of a cycle. One would be tempted to call the conjectured map $Z^i \longrightarrow H^i(X,\underline{K_i})$ the 'ith universal Chern class' map. It would be interesting to know what equivalence relation the map

$$\bigoplus_{htD = i-1} F_D^* \xrightarrow{\partial} Z^i$$

imposes on codimension i-cycles.

Since $E_1^{p,q} = 0$ if $-q < p$, one sees that 7.1) implies that $H^i(X,\underline{K}_i)$ are infinite cycles for the spectral sequence $'E_r^{p,q}$.

Remark 3. If n = 1 in 7.3), the exact sequence becomes

$$0 \longrightarrow \underline{K_1} \longrightarrow \underline{K_1(F)}_X \longrightarrow \underline{Z}^1 \longrightarrow 0 .$$

But $\underline{K_1} = \mathcal{O}_X^*$, so this exact sequence is

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \underline{F}_X^* \longrightarrow \underline{Z}^1 \longrightarrow 0 ,$$

defining Cartier (= Weil) divisors on the regular scheme X. In this case 7.5) becomes

$$F^* \longrightarrow Z^1 \xrightarrow{\partial} \text{Pic } X \longrightarrow 0$$

and the map $Z^1 \xrightarrow{\partial}$ Pic X is indeed the first Chern class, or line bundle, associated to a divisor.

Remark 4. If A is a regular local ring of dimension r, the working hypothesis in form 7.2 implies a surjection

7.6) $$\bigoplus_{\substack{\text{ht } D = i-1 \\ D \text{ prime in } A}} F_D^* \longrightarrow Z^i \longrightarrow 0$$

for each number i. Let $\underline{M}_A^i$ be the category of finitely generated A-modules M with $\text{codim}_{\text{SpecA}}(\text{support } M) \geq i$, and let $G_j(\underline{M}_A^i)$ (j=0,1) be the Grothendieck group of this abelian category (mod short exact sequences). Then $Z^i = G_0(\underline{M}_A^i/\underline{M}_A^{i+1})$ and one has

$$\bigoplus_{\substack{\text{ht } D = 1 \\ D \text{ prime in } A}} F_D^* = G_1(\underline{M}_A^i/\underline{M}_A^{i+1}) .$$

The K-theory exact sequence of a localization reads

$$\bigoplus_{\substack{\text{ht } D = i \\ D \text{ prime in } A}} F_D^* \longrightarrow G_0(\underline{M}_A^{i+1}) \longrightarrow G_0(\underline{M}_A^i) \longrightarrow Z^i \longrightarrow 0 .$$

The following result is elementary, and is proved by downward induction on i.

Proposition 7.7. The following statements are equivalent

1.) 7.6) is exact for all i.
2.) $G_0(\underline{M}_A^i) \cong Z^i$ for all i.
3.) $G_0(\underline{M}_A^{i+1}) \longrightarrow G_0(\underline{M}_A^i)$ is zero for all i.

In this connection one establishes easily

Proposition 7.8. $G_0(\underline{M}_A^i) \cong Z^i$ if either $i \leq 1$ or $r-i \leq 1$ ($r = \dim A$).

To produce some more evidence for 7.6 we introduce a concept

Definition 7.9. If A is a commutative noetherian domain, we say that A is clean if the natural map $G_0(\underline{M}_A^{i+1}) \longrightarrow G_0(\underline{M}_A^i)$ is zero for all i.

By proposition 7.7, the conjecture 7.6 is valid if and only if every regular local ring is clean. That there exist a vast collection of clean rings is a consequence of the next result.

Theorem 7.10. If A is clean, then so is every localization A_S of A. If A is clean and of finite Krull dimension, then so is A[t], the polynomial ring in one variable.

Proof. For localizations, one considers the commutative diagram

$$\begin{array}{ccc} G_0(\underline{M}_A^{i+1}) & \longrightarrow & G_0(\underline{M}_A^{\,i}) \\ \downarrow & & \downarrow \\ G_0(\underline{M}_{A_S}^{\,i+1}) & \longrightarrow & G_0(\underline{M}_{A_S}^{\,i}) \end{array} .$$

The case of a polynomial extension A[t] is more involved. We prove by descending induction on s that the map

$$G_0(\underline{M}^{s+1}_{A[t]}) \longrightarrow G_0(\underline{M}^{s}_{A[t]})$$

is zero (the induction starts with $s = \dim A + 1$). Supposing that this is the case for all $s > i$, we prove then that $G_0(\underline{M}^{i+1}_{A[t]}) \longrightarrow G_0(\underline{M}^{i}_{A[t]})$ is zero.

If P is a height i+1 prime in A[t], it suffices to prove that the class of the module A[t]/P maps to zero under $G_0(\underline{M}^{i+1}_{A[t]}) \longrightarrow G_0(\underline{M}^{i}_{A[t]})$. Observe that $\mathfrak{p} = P \cap A$ is either of height i or of height i+1 in A.

Case 1. $\mathrm{ht}_A \mathfrak{p} = i+1$. Then $P = \mathfrak{p}A[t]$ and $A[t]/P \cong A/\mathfrak{p}[t]$ as A[t]-modules. Consider the commutative diagram

$$\begin{array}{ccc} (A/\mathfrak{p}) \in G_0(\underline{M}_A^{i+1}) & \longrightarrow & G_0(\underline{M}^{i+1}_{A[t]}) \\ \downarrow & & \downarrow \\ G_0(\underline{M}_A^{i}) & \longrightarrow & G_0(\underline{M}^{i}_{A[t]}) \end{array}$$

The horizontal arrows are induced by

$$M \longmapsto A[t] \otimes_A M , \quad M \in \underline{M}_A .$$

The result follows by a diagram chase from the fact that A is clean.

Case 2. $ht_A \mathfrak{p} = i$. Let $A' = A/\mathfrak{p}$ and let P' be the image of P in $A'[t]$, so P' is a height 1 prime of $A'[t]$ with $P' \cap A' = (0)$. Let F' be the field of fractions of A'. Then $P'F'[t]$ is a principal ideal of $F'[t]$, say $P'F'[t] = (f'(t))$, with $f' \in F'[t]$. Choose $s \in A' - (0)$ so that $sf' \in A'[t]$ and let $f = sf'$. Since $F'[t]$ is a localization of $A'[t]$, one has

$$P' = (f')F'[t] \cap A'[t] = (fF'[t]) \cap A'[t]$$

and hence $fA'[t] \subset P'$.

Observe that $P'/fA'[t]$ is a torsion A'-module: for $P'F'[t] = fF'[t]$, so $(P'/fA'[t])_S = 0$ with $S = A' - (0)$. If now the prime $Q' \subset A'[t]$ contains an associated prime to $P'/fA'[t]$, then $F' \otimes_{A'} (A'[t]/Q') = 0$. Thus $Q' \cap A' \neq (0)$ and hence $ht_{A'} Q' \cap A' > 0$. Now lift Q' to a prime Q of $A[t]$. It follows that $ht_A Q \cap A > i$.

Lemma 7.11. The module $(A[t]/Q) = (A'[t]/Q')$ maps to zero under $G_0(\underline{M}^{i+1}_{A[t]}) \longrightarrow G_0(\underline{M}^{i}_{A[t]})$.

Proof. If $ht_{A[t]} Q = i+1$, then since $ht_A Q \cap A > i$, we have $ht_A Q \cap A = i+1$ and we are in case 1. If $ht_{A[t]} Q = s+1 > i+1$, then the induction hypothesis gives $(A[t]/Q)$ maps to zero under

$$G_0(\underline{M}^{s+1}_{A[t]}) \longrightarrow G_0(M^{s}_{A[t]}) ,$$

and hence to zero in $G_0(\underline{M}^{i}_{A})$ (since $i < s$). This completes the proof of the lemma.

Now $P'/fA'[t]$ has a filtration whose quotients are of the form $A'[t]/Q'$, where the prime Q' contains as associated prime of

$P'/fA'[t]$. By lemma 7.11, it follows that $P'/fA'[t]$ maps to zero under $G_0(M^{i+1}_{A[t]}) \longrightarrow G_0(M^i_{A[t]})$.

Consider now the short exact sequence of $A'[t]$ modules

7.12) $\quad 0 \longrightarrow P'/fA'[t] \longrightarrow A'[t]/(f) \longrightarrow A'[t]/P' \longrightarrow 0$.

Observe that $A'[t]/(f) = A[t]/(f_1,\mathfrak{p})$ where $f_1 \in A[t] - \mathfrak{p}$. Thus, all associated primes of $(f_1,\mathfrak{p})$ in $A[t]$ are of height $\geq$ i+1, and hence

$$A[t]/(f_1,\mathfrak{p}) \in \underline{M}^{i+1}_{A[t]} .$$

Lemma 7.13. $(A[t]/(f_1,\mathfrak{p}))$ maps to zero under $G_0(\underline{M}^{i+1}_{A[t]}) \longrightarrow G_0(\underline{M}^i_{A[t]})$.

Proof. Consider the exact sequence of $A[t]$ modules

$$0 \longrightarrow A[t]/\mathfrak{p}A[t] \xrightarrow{\cdot f_1} A[t]/\mathfrak{p}A[t] \longrightarrow A[t]/(f_1,\mathfrak{p}) \longrightarrow 0 .$$

Taking the Euler characteristic in $G_0(\underline{M}^i_{A[t]})$ gives the result.

Returning to 7.12), we see that $(P'/fA'[t])$ and $A'[t]/(f)$ both map to zero under $G_0(M^{i+1}_{A[t]}) \longrightarrow G_0(\underline{M}^i_{A[t]})$. It follows then that the same holds true for $A'[t]/P' \cong A[t]/P$. This completes the induction and hence the proof of 7.10 is complete.

Corollary 7.14. If k is a field, then all local rings of $\mathbb{P}^n_k$ are clean.

For they are obtained by localizing polynomial extensions of k.

Observe also that by 7.8 any regular local ring A of dimension ≤ 3 is clean, and hence so is any localization of a polynomial ring on A, by Theorem 7.10. One can also prove that a complete regular local ring of equicharacteristic is clean. A proof of the general assertion, that regular local rings are clean, eludes us at the moment.

Remark 5. The working hypothesis would follow if one could establish the map

$$G_n(\underline{M}_A^{i+1}) \longrightarrow G_n(\underline{M}_A^i)$$

is zero for all n and i, where A is a regular local ring and G_n is the higher Quillen K-theory [10] of an abelian category.

Remark 6. Using Grothendieck's theory of Chern classes in Hodge cohomology [8] §6. and the Universal property of $BG\ell(A)^+$, one can define "Hodge Chern classes"

$$K_i(A) \longrightarrow \Omega^i_{A/\mathbb{Z}}$$

where $\Omega^i_{A/\mathbb{Z}}$ is the module of Kähler differentials of the commutative ring A, as follows.

Theorem Grothendieck [8]. Suppose $\rho: G \longrightarrow \mathrm{Aut}_A E$ is a representation of the group G in the finitely generated projective A-module E. Then there is a unique family of classes

$$c_i(\rho) \in H^i(G,\Omega^i_{A/\mathbb{Z}}) \quad \text{(trivial action)}$$

subject to the following properties

Property 1: (Normalization) $c_0(\rho) = 1$ and

$$c_1(\rho) = \log(\det \rho) \in H^1(G,\Omega^1_{A/\mathbb{Z}}) = \mathrm{Hom}(G,\Omega^1_{A/\mathbb{Z}})$$

where $\log(\det \rho)(g) = \frac{d(\det \rho(g))}{\det \rho(g)}$, $g \in G$. Also, $c_i(\rho) = 0$ for $i > rk_A(E)$.

<u>Property 2</u>: (Naturality) If $\varphi: H \longrightarrow G$ is a homomorphism of groups and $f: A \longrightarrow B$ is a homomorphism of rings, then

$$c_i(\rho \circ \varphi) = \varphi^*(c_i(\rho)) \in H^i(H, \Omega^i_{A/\mathbb{Z}})$$

and
$$c_i(f_*(\rho)) = f_*(c_k(\rho)) \in H^i(G, \Omega^i_{B/\mathbb{Z}})$$

where $f_*(\rho)$ is the representation

$$G \xrightarrow{\rho} Aut_A(E) \longrightarrow Aut_B(B \otimes_A E)$$

and $f_*(c_i(\rho))$ is the image of $c_i(\rho)$ under the coefficient homomorphism

$$\Omega^i_{A/\mathbb{Z}} \longrightarrow \Omega^i_{B/\mathbb{Z}} .$$

<u>Property 3</u>: (Product formula) If

$$0 \longrightarrow (E', \rho') \longrightarrow (E, \rho) \longrightarrow (E'', \rho'') \longrightarrow 0$$

is a short exact sequence of representations, then $c(\rho) = c(\rho') \, c(\rho'')$, where

$$c(\rho) = \Sigma\, c_i(\rho) \in \bigoplus_{i \geq 0} H^i(G, \Omega^i_{A/\mathbb{Z}}) .$$

Here $\bigoplus_{i \geq 0} H^i(G, \Omega^i_{A/\mathbb{Z}})$ is a commutative ring, the multiplication induced from the ring structure on Kahler differentials.

Consider now the standard representations

$$\sigma_n: G\ell_n A \xrightarrow{1} G\ell_n A .$$

Then $$c_i(\sigma_n) \in H^i(G\ell_n(A), \Omega^i_{A/\mathbf{Z}}) .$$

The properties 1-3 imply that under restrictions

$H^i(G\ell_{n+1}(A), \Omega^i_{A/\mathbf{Z}}) \longrightarrow H^i(G\ell_n(A), \Omega^i_{A/\mathbf{Z}})$, we have $c_i(\sigma_{n+1}) \longmapsto c_i(\sigma_n)$.

But cohomology is representable, so

$$c_i(\sigma_n) \in [B_{G\ell_n(A)}, K(\Omega^i_{A/\mathbf{Z}}, i)] .$$

This sequence of homotopy classes of maps determines (many) maps $B_{G\ell(A)} \xrightarrow{f} K(\Omega^i_{A/\mathbf{Z}}, i)$ such that the following diagrams commute up to homotopy

$$\begin{array}{ccc} B_{G\ell_n(A)} & \xrightarrow{c_i(\sigma_n)} & K(\Omega^i_{A/\mathbf{Z}}, i) \\ \downarrow & \nearrow & \\ B_{G\ell(A)} & & \end{array}$$

Since $K(\Omega^i_{A/\mathbf{Z}}, i)$ is an H-space, the universal property of the map

$$B_{G\ell(A)} \longrightarrow B^+_{G\ell(A)}$$

(see [6], Theorem 2.5)

determines a factorization

$$\begin{array}{ccc} B_{G\ell(A)} & \xrightarrow{f} & K(\Omega^i_{A/\mathbf{Z}}, i) \\ \downarrow & \nearrow f^+ & \\ B^+_{G\ell(A)} & & \end{array}$$

The effect of f^+ on homotopy,

$$K_i(A) \longrightarrow \Omega^i_{A/\mathbf{Z}} , \quad i \geq 0 ,$$

is the desired Hodge Chern class.

Sheafifying, this gives a map $\underline{K}_i \longrightarrow \underline{\Omega}^i_{X/\mathbf{Z}}$ of sheaves on X, and hence homomorphisms

7.15) $$H^i(X,\underline{K}_i) \longrightarrow H^i(X,\underline{\Omega}^i_{X/\mathbb{Z}})$$

The composition of the conjectured surjection

$$Z^i \longrightarrow H^i(X,\underline{K}_i)$$

of 7.5) with the map 7.9) would be the "cycle map" $Z^i \longrightarrow H^i(X,\Omega^i_{X/\mathbb{Z}})$ from codimension i-cycles to Hodge cohomology of X. Thus, 7.5) implies a particularly attractive interpretation of the cycle map

$$Z^i \longrightarrow H^i(X,\Omega^i_{X/\mathbb{Z}}) ;$$

namely it would be the composition of the universal i^{th} Chern class map

$$Z^i \twoheadrightarrow H^i(X,\underline{K}_i)$$

with the "Hodge Chern class" map

$$H^i(X,\underline{K}_i) \longrightarrow H^i(X,\Omega^i_{X/\mathbf{Z}}) .$$

October 31, 1972

Addendum

A. Dress has brought to my attention a result of Swan*):

Theorem. Let A be a semilocal Dedekind ring with field of fractions F. Then for any finite group G, the map

$$K_0(\underline{P}_A(G)) \longrightarrow K_0(\underline{P}_F(G))$$

is an isomorphism.

Using this result and the method of §3. we deduce

Theorem A1. Suppose that A is a semi local Dedekind ring such that for all $\underline{m} \in \max A$, $A/\underline{m}$ is finite. Then for any group G, the map

$$\coprod_{m \in \max A} K_0(\underline{P}_{A/\underline{m}}(G)) \longrightarrow K_0(\underline{P}_A(G)) ,$$

of Theorem 3.1 is zero.

Consequently we deduce

Theorem A2. If A is a semi local Dedekind ring such that $A/\underline{m}$ is finite for all $\underline{m} \in \max A$, then one has the short exact localization sequences

$$0 \longrightarrow K_n(A) \longrightarrow K_n(F) \longrightarrow \coprod_{\underline{m} \in \max A} K_{n-1}(A/\underline{m}) \longrightarrow 0 ,$$

where F is the field of fractions of A.

As in Theorem 4.3 we deduce that $K_{4n+3}(R)$ is finite of order independent of R for all semi local subrings of the rational numbers.

*) R. G. Swan, The Grothendieck ring of a finite group, Topology 2 (1963), Theorem 3 p. 87.

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