# Problems about Higher K-functors

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I have attempted to restrict this collection of research problems to "well posed" problems. This has necessitated some drastic revision or even omission of a number of otherwise very interesting problems that were submitted. Several other problems have been omitted since I consider them solved already. I have not attempted to trace the history of each problem. A credit of a problem to one author means then that I have adopted that author's formulation of the problem. With the exception of Problem 1, I have not included Lichtenbaum's conjectures [20] in this set, since they are amply discussed in his report.

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Problem 1: Compute $K_3(\mathbb{Z})$. Lichtenbaum [20] predicted that $\#(K_3(\mathbb{Z})) = 24$ and further that $K_3(R) \longrightarrow K_3(Q)$ is an isomorphism for all subrings R of Q. The latter is known when R is semilocal [10]. The work of Quillen [16] shows that the homomorphism $\pi_3^s \longrightarrow K_3(\mathbb{Z})$ contributes at least a cyclic subgroup of order 24 to $K_3(\mathbb{Z})$. Karoubi has indicated that his L-theory can be used to

*) R. G. Swan, Problems about higher K-functors. Parts of this discussion have been edited and included here with the author's permission.

show $\#(K_3(\mathbb{Z})) > 24$, and Lichtenbaum has shown that the latter implies that $\#(K_3(\mathbb{Z}))$ is then even divisible by $192 = 24\cdot 8$. (S.M.G.)

Problem 2: Stability for $K_i$ $(i \geq 1)$: For $n \geq 3$ define $BG\ell_n(A) \longrightarrow BG\ell_n(A)^+$ to be the acyclic map corresponding to the perfect subgroup $E_n'(A)$ of $\pi_1(BG\ell_n(A))$ (for $n \geq 3$) where $E_n'(A)$ is the normal subgroup generated by $E_n(A)$. Then $K_i(A) = \lim_{n\to\infty} \pi_i(BG\ell_n(A)^+)$. Suppose that A is an algebra, finitely generated as a module, over a commutative ring k with max(k) a noetherian space of dimension d. Show that the map $s_n\colon \pi_i(BG\ell_n(A)^+) \longrightarrow \pi_i(BG\ell_{n+1}(A)^+)$ (a) is surjective for $n \geq d+i$ and (b) is injective for $n > d+i$. This is known for $i = 1$ (Bass [2] and Wasserstein [18]) and (a) is known for $i = 2$ (Dennis [6]). (H.B.)

Problem 3: The "fundamental theorem" of K-theory: For a ring R let P(R) be the category of finitely generated projective modules over R and let Nil (R) be the category shose objects are pairs $(P,\nu)$, $P \in$ P(R), and $\nu\colon P \longrightarrow P$ a nilpotent endomorphism of P. [2, page 652]. Then P(R) is a retract of Nil (R) by $(P,\nu) \longmapsto P$, $P \longmapsto (P,0)$. Thus, if we consider the Quillen K-groups of categories with exact sequences [15] we have $K_n(\text{Nil}\ (R)) = K_n(R) \oplus \text{Nil}_n(R)$, where $\text{Nil}_n(R)$ is the kernel of $K_n(\text{Nil}\ (R)) \longrightarrow K_n(R)$. Show that, in analogy with results for $K_1$ [2],

(1) $K_n(R[t]) = K_n(R) \oplus Nil_{n-1}(R)$ and

(2) $K_n(R[t,t^{-1}]) = K_n(R) \oplus K_{n-1}(R) \oplus Nil_{n-1}(R) \oplus Nil_{n-1}(R)$.

As a consequence, one would have the contracted functor exact sequence

$$0 \longrightarrow K_n(R) \longrightarrow K_n(R[t]) \oplus K_n(R[t^{-1}] \longrightarrow K_n(R[t,t^{-1}]) \longrightarrow K_{n-1}(R) \longrightarrow 0.$$

with a natural splitting. (R.G.S.)

Problem 4: Localization Sequence:

As a means of attempting problem 3, one might try to extend the K-theory localization sequence. Let $\underline{H}(R)$ be the category of R-modules M which have a finite resolution $0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ with $P_i \in \underline{P}(R)$. If S is a central multiplicative subset of R, let $\underline{H}_S(R)$ be the full subcategory of $M \in \underline{H}(R)$ with $M_S = 0$. If S consists of non zero divisors, show that there is an exact sequence

$$\ldots \longrightarrow K_n(\underline{H}_S(R)) \longrightarrow K_n(R) \longrightarrow K_n(R_S) \longrightarrow H_{n-1}(\underline{H}_S(R)) \longrightarrow ,$$

where the map $K_n(\underline{H}_S(R)) \longrightarrow K_n(R)$ is induced by the inclusion $\underline{H}_S(R) \hookrightarrow \underline{H}(R)$. If R is left regular (i.e., left noetherian and each finitely generated left R-module is in $\underline{H}(R)$) then this results from work of Quillen [15] without any hypothesis on S. In general, the hypothesis on S is essential.

Remark. Murthy has pointed out a consequence of the localization sequence above: if a and b are central in R, if R = Ra+Rb, and if a is a non zero divisor in R, then there is an exact Mayer-Vietoris sequence

$$\cdots \longrightarrow K_n(R) \longrightarrow K_n(R_a) \oplus K_n(R_b) \longrightarrow K_n(R_{ab}) \longrightarrow K_{n-1}(R) \longrightarrow \cdots .$$

This follows from comparing two localization sequences, using the fact that $\underline{H}_a(R) = \underline{H}_a(R_b)$. The Mayer-Vietoris sequence has been obtained by Gersten [8] under quite different hypotheses. (R.G.S.)

Problem 5: Define functors $\Omega$,S on an appropriate category of categories such that $K_n(\Omega\mathcal{C}) = K_{n+1}(\mathcal{C})$ and $K_n(S\mathcal{C}) = K_{n-1}(\mathcal{C})$ for all n. If F: $\mathfrak{A} \longrightarrow \mathfrak{B}$ is an exact functor, define, in a functorial way, a category $\mathfrak{F}$ (the "fibre" of F) and exact functors i: $\mathfrak{F} \longrightarrow \mathfrak{A}$, j: $\mathfrak{B} \longrightarrow \Omega\mathfrak{F}$ (respt. $S\mathfrak{B} \longrightarrow \mathfrak{F}$) such that $BQ(\mathfrak{F})$ is the fibre of the map $BQ(\mathfrak{A}) \longrightarrow BQ(\mathfrak{B})$ and such that

$$\cdots \longrightarrow K_n(\mathfrak{F}) \xrightarrow{i_*} K_n(\mathfrak{A}) \xrightarrow{F_*} K_n(\mathfrak{B}) \xrightarrow{j_*} K_{n-1}(\mathfrak{F}) \longrightarrow \cdots$$

is its homotopy sequence. Here $Q(\mathfrak{A})$ is Quillen's categorical construction [15] and $BQ(\mathfrak{A})$ is the geometric realization of the nerve of $Q(\mathfrak{A})$. For this to be really useful, one would like to reduce all problems about higher K's (like problem 4) to the case where only $K_0$ and $K_1$ are involved, by means of $\Omega$ and S.

In the special case of abelian categories, we are led to pose the following question. If $\mathfrak{A}$ is an abelian category, is there an abelian category $\mathfrak{B}$ with a Serre subcategory $\mathfrak{S}$ such that $\mathfrak{B}/\mathfrak{S} \approx \mathfrak{A}$ and $K_n\mathfrak{B} = 0$ for $n \geq 1$ ? If $\mathfrak{B}$ and $\mathfrak{S}$ can be constructed functorially in $\mathfrak{A}$, then $\mathfrak{S}$ would be a reasonable candidate for $\Omega\mathfrak{A}$.

Closely related to the questions above are the following problems. Can one give an axiomatic characterization of Quillen's functors $K_n$ for categories with exact sequences? Can one define $K_n(A)$ for $n < 0$ for categories with exact sequences so that the results proved by Quillen [15] continue to hold in regative dimensions, and such that $K_n(\underline{P}(R)) = K_n(R)$ in the sense of Bass [2] and Karoubi-Villamayor [12] for $n < 0$ ? (R.G.S.)

Problem 6: Let $\mathcal{A}$ be an admissible subcategory of an abelian category. Let $\mathcal{B}$ be a full subcategory of $\mathcal{A}$ which is closed under direct sums and is cofinal in the sense that every object of $\mathcal{A}$ is a direct summand of an object of $\mathcal{B}$. Assume that $\mathcal{B}$ is closed in $\mathcal{A}$ under at least one of the following: extensions, kernels of epimorphisms, or cokernels of monomorphisms. Is it true that $K_n(\mathcal{B}) \longrightarrow K_n(\mathcal{A})$ is injective? This is the case if $n = 0$. The problem is intended mainly as a test for the ideas mentioned in problem 5. It may also be of use in connection with problem 4: If $\mathcal{A}$ is the subcategory of $\underline{H}_S(R)$ of modules with projective dimension 1, and if $\mathcal{B}$ is the category of such modules M with a resolution $0 \longrightarrow R^n \longrightarrow R^n \longrightarrow M \longrightarrow 0$, then the conjecture above applies, $K_n(\mathcal{A}) = K_n(\underline{H}_S(R))$, and it is reasonable to conjecture that

$$0 \longrightarrow K_n(\mathcal{B}) \longrightarrow K_n(\underline{H}_S(R)) \longrightarrow K_n(R) \longrightarrow K_n(R_S)$$

is exact. This is true when $n = 0$. (R.G.S.)

Problem 7: It is known that $K_3(A) = H_3(St(A), \mathbb{Z})$ ([9], Theorem 2.22) Can one develop methods to compute $H_3(St(A), \mathbb{Z})$, along the lines developed to compute $K_2(A) = H_2(E(A), \mathbb{Z})$? As a first step give a good "diagram theoretic" interpretation of $H_3(G, \mathbb{Z})$ when G is a group satisfying $H_1(G) = H_2(G) = 0$. (S.M.G.)

Problem 8: It is now known that $K_n(\mathbb{Z}\{X\}) = K_n(\mathbb{Z})$, [9], if $\mathbb{Z}\{X\}$ is the free associative algebra on the set X. Anderson [1] has used this fact to prove that $K_n(A) = K_n^S(A)$ for all rings A. He writes that if $K_*^V(\mathbb{Z}\{X\}) = K_*^V(\mathbb{Z})$, then $K_* = K_*^V$, where $K_*^V$ denotes the Volodin K-theory [19]. He asks then, is $K_*^V(\mathbb{Z}\{X\}) = K_*^V(\mathbb{Z})$ ?

(D.W.A.)

Problem 9: Suppose A is a discrete valuation rign with maximal ideal $\underline{m}$. Then is the transfer map [15] $K_n(A/\underline{m}) \longrightarrow K_n(A)$ always zero? This is true if $n \le 2$ by Dennis-Stein [7]. It is true for general n, if $A/\underline{m}$ is finite or if A is a k-algebra (k a field) and $A/\underline{m}$ is a finite separable extension of k by Gersten [10]. In fact we suspect that if F is the field of fractions of A, then every finite subcomplex of the fibre of the map $B_{G\ell A}^+ \longrightarrow B_{G\ell F}^+$ is contractible in the total space. This is the case in the situations dealt with in [10]. (S.M.G.)

Problem 10: Suppose that X is a regular variety over an algebraically closed field k of characteristic 0. The sheaf $\underline{\underline{K}}_n$ of abelian groups is then constructed by sheafifying the presheaf

$U \longmapsto K_n(\Gamma(U,\mathcal{O}_X))$. We would like to know whether $\underline{K}_n$ is a Cohen-Macauley sheaf [11], p. 238. This is in fact the case if X is a curve or if n = 1. (S.M.G.)

Problem 11: In analogy to Hodge cohomology, let $H_K^{ij}(X) = H^i(X,\underline{K}_j)$ if X is a regular scheme. By a result of Brown and Gersten [5] there is a spectral sequence

$$E_2^{p,q} = H_K^{p,-q}(X) \Rightarrow K_{-p-q}(X) \quad ,$$

if X has finite (Krull) dimension. Is there a Dold-Thom isomorphism theorem for the $H_K^{**}$- theory? That is, suppose that E is a vector bundle on X of rank r. Then is $H_K^{**}(\mathbb{P}(E))$ a free module over $H_K^{**}(X)$ of rank r, with free basis $1,c_1,\ldots,c_1^{r-1}$, where $c_1 \in H_K^{1,1}(\mathbb{P}(E)) = \mathrm{Pic}(\mathbb{P}(E))$ is the class of $\mathcal{O}_{\mathbb{P}(E)}(-1)$? Is the $H_K^{**}$- theory polynomial invariant? Namely, is $H^i(X,\underline{K}_j)$ $= H^i(X[t],\underline{K}_j)$ where $X[t] = X \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{Z}[t]$ ? (S.M.G.)

Problem 12: In the notation of problem 11 , is $H^p(X,\underline{K}_q) = 0$ if $p > q$? This is the case if q = 1, since the sheaf $\underline{K}_1 = \mathcal{O}_X^*$ is Cohen-Macauley (X is regular). (S.B.)

Problem 13: Borel has shown that if $\mathcal{O}$ is the ring of integers in the number field F which has r real places and c complex places, then rank $K_{4n+1}(\mathcal{O}) = r+c$ $(n > 0)$ and rank $K_{4n+3}(\mathcal{O}) = c$. This phenomenon should be explained by a theory of characteristic classes, to be evaluated on $K_*(\mathbb{R})$ or $K_*(\mathbb{C})$ for the imbeddings of $\mathcal{O}$ in $\mathbb{R}$

or in $\mathbb{C}$. Quillen has suggested that Deligne's refinement of DeRham cohomology is the target theory for a suitable theory of Deligne-Chern classes of group representations, which would extend to give the desired theory of K-theory characteristic classes (compare the argument in §7, [10]). It is important that such a functorial definition of Borel's classes be given in understanding the higher regulators of a number field. (S.M.G.)

Problem 14: Quillen has recently proposed a definition of $K_n(\mathcal{A})$ where $\mathcal{A}$ is an abelian category. $K_0(\mathcal{A})$ is the usual Grothendieck group with relations all short exact sequences. However, $K_1(\mathcal{A})$ is not the group that appears on Bass' book (see [9], §5). Can one give a more algebraic description of $K_1(\mathcal{A})$ in terms of the category $\mathcal{A}$, than that given by Quillen, $K_1\mathcal{A} = \pi_2(|NQ(\mathcal{A})|)$ ? For example, (see [9], §5) if X is a complete regular curve over the algebraically closed field k, then $K_1(X)$ is defined to be $K_1$ of the category of coherent sheaves of $\mathcal{O}_X$-modules. Then $K_1(X) \cong k^* \oplus SK_1(X)$, where $SK_1(X) = (k^* \otimes_{\mathbf{Z}} D)/R$. Here D is the divisor group (free abelian group on closed points P of X) and R is the subgroup generated by all sums

$$\sum_P \lambda_P\{f,g\} \otimes P,$$

where f and g are non zero rational functions and

$$\lambda_P\{f,g\} = (-1)^{\mathrm{ord}_P(f)\cdot \mathrm{ord}_P(g)} \cdot \frac{f^{\mathrm{ord}_P(g)}}{g^{\mathrm{ord}_P(f)}}(P) \; . \qquad \text{(S.M.G.)}$$

Problem 15: Lichtenbaum asks whether $K_n(k)$ is divisible if k is algebraically closed ($n > 0$), with torsion subgroup zero if n is even and isomorphic to $W_{(\frac{n+1}{2})}$ if n is odd. One may observe in this connection that $K_2(k)$ is uniquely divisible.

(S.L.)

Problem 16: Let F be a field, $F_s$ its separable closure, and $G = Gal(F_s/F)$. Does there exist a spectral sequence

$$E_2^{p,q} = H^p(G, K_{-q}(F_s))$$

which, when it converges, filters the K-theory of F (for $p+q \leqslant 0$)? More generally, one can formulate analogous questions for K-theory of unramified (étale) extensions of rings. (S.L.)

Problem 17: Can there exist p-torsion in $K_n(F)$, if the field F has characteristic p ? The frobenius map shows this is not the case if F is perfect. (S.U.C.)

Problem 18: If A is a ring of characteristic p, then is $Ker(K_n(A[t]) \longrightarrow K_n(A))$ always a p-torsion group ? (S.U.C.)

Problem 19: Can one describe $K_n$ of a field by generators and relations (cf. Matsumoto's presentation of $K_2$ [13])? What about division rings? (R.G.S.)

Problem 20: Compute $K_n(R/I)$ where R is the ring of integers of a number field and I is a non zero ideal (solved by Dennis and Stein [7] for $K_2$). What about $K_n$ of finite rings in general? (R.G.S.)

Problem 21: Let R be an order over $\mathbb{Z}$ in a semisimple $\mathbb{Q}$-algebra. Is $K_n(R)$ finitely generated? What is its rank? Similar questions for $G_n(R)$. Of special interest are group rings and maximal orders. These questions have been answered by Borel [4] and Quillen [14] for the ring of integers of a number field. (R.G.S.)

Problem 22: What is the relation between $K_n(R)$ and $K_n(R/I)$ where I is a nilpotent ideal? For a special case see [3]. Another special case is the following. If I is any abelian group, make I a ring by $I^2 = 0$, adjoin a unit getting $K^+ = \mathbb{Z} \times I$ with the obvious multiplication. Compute then $K_n(I^+)$. (R.G.S.)

Problem 23: If $F_1$ and $F_2$ are function fields of algebraic curve $C_1$ and $C_2$ defined over the field k, then $K_0(F_1 \otimes_k F_2)$ is the ring of classes of correspondences between $C_1$ and $C_2$. This can be used to give a simplified version of Raquette's proof of the Riemann hypothesis for curves [17]. What is the geometric meaning of $K_n(F_1 \otimes_k F_2)$ and more generally of $K_n(F_1 \otimes_k \cdots \otimes_k F_r)$ ? (R.G.S.)

<u>Problem 24</u>: Group rings of free products:

There is interest in extending the results of Stallings and Gersten on $K_1$ and $K_0$ of free products [2, page 697] to the higher Quillen K-functors. Let us say that a ring R is (left- ) <u>super-coherent</u> if every polynomial extension $\Lambda$ of R is (left- ) coherent. In particular, left noetherian rings are supercoherent. We have established the following

<u>Theorem</u>: If A and B are groups (or more generally monoids) and R is a noetherian regular ring such that R[A] and R[B] are supercoherent and regular*), then there are exact sequences for all $n \geq 0$

$$0 \longrightarrow K_n(R) \longrightarrow K_n(R[A]) \oplus K_n(R[B]) \longrightarrow K_n(R[A*B]) \longrightarrow 0 .$$

We pose the following problem. Suppose that $D = A \underset{C}{*} B$ is the free product of groups A and B amalgamating the common subgroup C. Suppose R is a ring such that R[C] is noetherian and regular, and such that R[A] and R[B] are both supercoherent and ular. Then establish a long Mayer-Vietoris sequence

$$\cdots \to K_{n+1}(R[D]) \to K_n(R[C]) \to K_n(R[A]) \oplus K_n(R[B]) \to K_n(R[D]) \to \cdots$$

Note that the theorem above is a special case of the conjecture, when C = (1). We do not know whether the hypotheses of the theorem or conjecture can be weakened. The proof of the theorem makes use of all assumptions. It depends on the result of

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*) A (non notherian) ring $\Lambda$ is said to be (left) regular if every $\Lambda$-module of <u>finite presentation</u> has finite projective dimension.

Choo, Lam, and Luft [21] that states that R[D] is supercoherent under these hypotheses (regularity of R[D] is also readily established by these methods). An interesting preprint of Waldhausen [22] also has bearing on this problem.

Similarly, one should state the analogous conjecture for Higman-Neumann-Neumann (HNN- ) extensions (see Miller [23] for definitions). Suppose that $C \overset{\alpha}{\underset{\beta}{\rightrightarrows}} A$ are two injections of the group C into the group A, and $D = HNN(A,C;\alpha,\beta)$. Suppose that R[C] is noetherian and regular and R[A] is supercoherent and regular. Then establish the long exact sequence

$$\cdots \to K_{n+1}(R[D]) \to K_n(R[C]) \xrightarrow{\alpha_* - \beta_*} K_n(R[A]) \to K_n(R[D]) \to \cdots$$

Here also we do not know whether the hypotheses are essential. However, Waldhausen states that, for $R = \mathbb{Z}$, the methods of his preprint [22] can be used to show that $\mathbb{Z}[D]$ is coherent and regular, under the hypotheses above.

(S.M.G.)

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