Higher K-theory of Rings

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In this article we shall summarize several of the proposed K-theories of rings and attempt to collect information about each theory and relate the theories where possible. The subject is very much in a state of flux and the notation has not yet been established, so we have chosen notations we find convenient. There are two other survey articles [12],[49] in the subject which have considerable overlap with this article. We have decided to restrict attention to linear theories, that is algebraic analogs of the homotopy of the general linear group of a ring. Even with this restricted aim, we have not included important topics. For a more complete account of the Anderson-Segal theory, see Anderson's talk. For an account of the Volodin theory [57], see Wagoner's talk. And for the explicit computations of higher K-groups, see Quillen's talk. Swan [49] gives a more complete account of the history of the subject, but we have described the minimum history we think is needed to motivate the higher K's.

Contents

§1. Early progress towards a K-theory of rings.

Algebraic K-theory arises from Grothendieck's proof of the Riemann-Roch Theorem [7]. A construction of the following type was introduced. Suppose that $\mathcal{C}$ is a category and $\mathcal{E}$ is a collection of diagrams $\underline{e}$ of $\mathcal{C}$ of the form

$$\underline{e}: \; E' \longrightarrow E \longrightarrow E'' \;.$$

Then the Grothendieck group $K_0(\mathcal{C},\mathcal{E})$ is an abelian group*), universal for mappings φ: obj $\mathcal{C} \longrightarrow A$, where A is an abelian group, such that

$$\varphi(E) = \varphi(E') + \varphi(E'')$$

where the diagram $\underline{e} = (E' \longrightarrow E \longrightarrow E'')$ is in $\mathcal{E}$. For example, if $\mathcal{C}$ is an abelian category and $\mathcal{E}$ is the class of all short exact sequences, then $K_0(\mathcal{C},\mathcal{E})$ is denoted $K_0(\mathcal{C})$ and is called the Grothendieck group of $\mathcal{C}$. If $\mathcal{P}_A$ is the category of projective left A-modules of finite type (A a unitary associative ring) and $\mathcal{E}$ is the class of short exact sequences of $\mathcal{C}$, then $K_0(\mathcal{P}_A)$ is denoted $K_0(A)$. One observes that $K_0(A)$ is a covariant functor of A, for a ring homomorphism

$$f: A \longrightarrow B$$

induces a functor $\mathcal{P}_A \longrightarrow \mathcal{P}_B$ by $P \longrightarrow B \otimes_A P$.

If X is a CW complex, and $\mathcal{C}$ is the category of complex vector bundles on X and $\mathcal{E}$ is the class of short exact sequences of vector bundles, then $K_0(\mathcal{C},\mathcal{E})$ is denoted $K^0(X)$. It was proved by Atiyah and Hirzebruch [2] that $K^0(X)$ is one functor of a

*) Clearly $K_0(\mathcal{C},\mathcal{E})$ exists if $\mathcal{C}$ admits a small equivalent subcategory.

generalized cohomology theory of complexes, which has since proved crucial in settling long open questions of topology. A result of Serre [47], greatly generalized by Swan [53], provided a dictionary for translating topological notions into algebraic notions:

Theorem 1.1 (Swan). Let X be a compact Hausdorff space and let C(X) be the ring of continuous complex valued functions on X. Then there is an equivalence between the category of complex vector bundles B on X and the category of finitely generated projective C(X) modules given by $B \longmapsto \Gamma(B)$, the sections of the bundle B viewed as a C(X) module.

The next step was taken by Bass [4] who proposed a candidate for $K_1(A)$ of a ring A, using the dictionary relating algebraic to topological notions. Let $G\ell_n(A)$ imbed in $G\ell_{n+1}(A)$ by

$$M \longmapsto \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$$

and let $G\ell(A) = \varinjlim_n G\ell_n(A)$. Then set $K_1(A) = H_1(G\ell(A),\mathbb{Z}) = G\ell(A)^{ab}$, the commutator factor group of $G\ell(A)$. A result of J.H.C. Whitehead [35] states that the commutator subgroup $G\ell(A)'$ is generated by elementary matrices, $E_{ij}(a)$, $i\neq j$, $a\in A$, where $E_{ij}(a)$ is 1 on the diagonal and 0 off the diagonal, except at the ij position where one puts $a\in A$. These elementary matrices then are to be considered "null homotopies" in $G\ell(A)$. (One can put in a parameter E_{ij} (at) and let $t \to 0$, connecting $E_{ij}(a)$ to 1, provided this makes sense. For example, if A is a Banach algebra over $\mathbb{R}$). It was clear from the outset that an alternative candidate for K_1 existed, $K_1^h(A) = G\ell(A)/UP(A)$, where UP(A) is the subgroup generated by unipotent matrices $M = 1 + N$, N nilpotent. (One can put in a parameter again $1 + Nt$ and let $t \to 0$, provided this makes sense). The functor $K_1^h(A)$ is a homotopy functor in the following sense.

Definition: Two ring homomorphisms $A \overset{f_0}{\underset{f_1}{\rightrightarrows}} B$ are homotopic if there is a homomorphism $A \xrightarrow{F} B[t]$ such that $(t \to i) \cdot F = f_i$, $i = 0,1$. Here $t \to i$ is the map $B[t] \longrightarrow B$ given by evaluating t at i. The functor K: Ring $\longrightarrow$ Ab is a homotopy functor if $K(f_0) = K(f_1)$ whenever f_0 and f_1 are homotopic.

These definitions were justified by a spate of exact sequences (see [49] for more complete information). We mention one because of its influence on the further development of the subject.

Theorem 1.2 (Milnor [35]). Given a cartesian diagram of rings

$$\begin{array}{ccc} D & \longrightarrow & A \\ \downarrow & & \downarrow f \\ B & \xrightarrow{g} & C \end{array}$$

where f is surjective, then there is an exact sequence of "Mayer-Vietoris type"

$$K_1(D) \longrightarrow K_1(A) \oplus K_1(B) \longrightarrow K_1(C) \longrightarrow K_0(D) \longrightarrow K_0(A) \oplus K_0(B) \longrightarrow K_0(C)$$

The functors K_1 and K_0 have played an important role in algebra which is beyond the scope of this article. We mention several results, because of their fundamental significance.

Theorem 1.3. If A is noetherian and left regular (that is, each A module of finite type has finite projective dimension), then $K_i(A) \simeq K_i(A[t])$ for i = 0,1.

The result for i = 0 is due to Grothendieck and for i = 1 to Bass, Heller, and Swan [6]. In this connection, it is worth mentioning

Theorem 1.4 [13]. If $K_1(R) \longrightarrow K_1(R[t])$ is an isomorphism, then $K_1(R) \longrightarrow K_1(R\{X\})$ is also an isomorphism. This is in fact the case if R left regular. Here $R\{X\}$ is the free associative algebra on the set X over R.

Theorem 1.5 [3]. If R is any ring, then there is a natural exact sequence

$$0 \longrightarrow K_1(R) \longrightarrow K_1(R[t]) \oplus K_1(R[t^{-1}]) \longrightarrow K_1(R[t,t^{-1}]) \longrightarrow K_0(R) \longrightarrow 0 .$$

Consequently, if R is left regular, then

$$K_1(R[t,t^{-1}]) \simeq K_1(R) \oplus K_0(R) .$$

A decisive step was taken by Milnor [35] in defining higher K's. Motivated by a paper of R. Steinberg [48] who had defined covering groups of algebraic groups, Milnor defined the Steinberg group St(R) of a ring R to be generated by symbols $x_{ij}(r)$, $r \in R$, $i \neq j$; $i,j \geq 1$, and subject only to three types of relations

1.) $x_{ij}(r)x_{ij}(r') = x_{ij}(r+r')$

2.) $[x_{ij}(r), x_{jk}(r')] = x_{ik}(rr')$; $i \neq k$

3.) $[x_{ij}(r), x_{k\ell}(r')] = 1$; $i \neq \ell$, $j \neq k$.

There is a canonical surjection $St(R) \xrightarrow{\varphi} \mathcal{E}(R)$ given by

$x_{ij}(r) \longmapsto E_{ij}(r)$. Milnor proved [35] that the kernel of φ is precisely the center of St(R), and defined

$$K_2(R) = \text{Ker } \varphi .$$

Hence there is an exact sequence

$$1 \longrightarrow K_2R \longrightarrow St(R) \longrightarrow G\ell(R) \longrightarrow K_1(R) \longrightarrow 0 .$$

In fact, the extension

1.6) $$1 \longrightarrow K_2(R) \longrightarrow St(R) \longrightarrow \mathcal{E}(R) \longrightarrow 1$$

was shown by Kervaire [29] to be the universal central extension of the perfect group $\mathcal{E}(R)$. It follows then that

1.7) $$K_2(R) = H_2(\mathcal{E}(R),\mathbb{Z}) .$$

Milnor also extended his exact sequence 1.2. He proved that, if both f and g are surjective, then the sequence continues

1.8) $$K_2D \longrightarrow K_2(A)\oplus K_2(B) \longrightarrow K_2(C) \longrightarrow K_1(D) \longrightarrow K_1(A)\oplus K_1B) \longrightarrow \dots .$$

At this point one may observe that $B_{\mathcal{E}(R)}$ is the fibre of the map $B_{G\ell(R)} \longrightarrow B_{K_1(R)}$ induced by $G\ell(R) \longrightarrow K_1(R)$ (and B_G is an Eilenberg-MacLane space of type K(G,1), G a group). Also, $B_{St(R)}$ is the homotopy fibre of the map $B_{\mathcal{E}(R)} \longrightarrow K(K_2(R),2)$ representing the extension 1.6) in $H^2(\mathcal{E}(R),K_2(R)) = H^2(B_{\mathcal{E}(R)},K_2(R))$ $= [B_{\mathcal{E}(R)},K(K_2(R),2)]$. This suggests [54,p. 207] that $K_3(R)$ $= H_3(St(R),\mathbb{Z})$ and higher K's should arise from a Postnikov type tower for $B_{G\ell(R)}$. We shall return to this question later, when the Quillen K-theory has been described.

§2. Quillen K-theory.

From his work on the Adams conjecture [37], and in particular his computation of the fibre of the Adams operation ψ^q-1: BU ⟶ BU, Quillen was motivated to propose an extremely elegant definition of higher K-groups. Since he will describe this in his talk, we recall only a basic consequence of his work.

Theorem 2.1 (Quillen). If X is a based CW complex and E is a perfect normal subgroup of $\pi_1 X$, then there is a map $X \xrightarrow{f} X^+$, unique up to homotopy, such that E is the kernel of $\pi_1(f)$ and such that the homotopy fibre F of f has the same integral homology as a point.

If one applies this to the space $B_{G\ell(R)}$ and subgroup $\mathcal{E}(R)$ of $G\ell(R)$, one gets a map $B_{G\ell(R)} \xrightarrow{f} B^+_{G\ell(R)}$ where $B^+_{G\ell(R)}$ has the same integral homology as $B_{G\ell}(R)$, and $\pi_1 B^+_{G\ell(R)} = G\ell(R)/\mathcal{E}(R) = K_1(R)$.

Proposition 2.2: $\pi_2 B^+_{G\ell(R)} = K_2(R)$.

Proof: Let F be the fibre of the map $B_{G\ell(R)} \longrightarrow B^+_{G\ell(R)}$. Since $H_*(F,\mathbf{Z}) = H_*(pt,\mathbf{Z})$, we have $H_i(F,\mathbf{Z}) = 0$, $i > 0$. But the low dimensional terms exact sequence associated to the map $F \longrightarrow K(\pi_1(F),1)$ shows that $H_1(\pi_1(F),\mathbf{Z}) = H_2(\pi_1(F),\mathbf{Z}) = 0$. The homotopy exact sequence of the fibration $F \longrightarrow B_{G\ell(R)} \longrightarrow B^+_{G\ell(R)}$ gives the exact sequence

$$0 \longrightarrow \pi_2 B^+_{G\ell(R)} \xrightarrow{\partial} \pi_1(F) \longrightarrow G\ell(R) \longrightarrow K_1(R) \longrightarrow 0 ,$$

whence the extension

2.3) $$0 \longrightarrow \pi_2 B^+_{G\ell(R)} \xrightarrow{\partial} \pi_1(F) \longrightarrow \mathcal{E}(R) \longrightarrow 0 .$$

A general property of the homotopy exact sequence is that the image of the boundary map $\pi_2 B \longrightarrow \pi_1 F$ of a fibration $F \longrightarrow E \longrightarrow B$ is always central in $\pi_1 F$. Thus 2.3) is a central extension. Furthermore, since $H_1(\pi_1(F),\mathbb{Z}) = H_2(\pi_1(F)\mathbb{Z}) = 0$, it follows from a lemma of Kervaire ([29], lemma 2, p. 215) that 2.3) is the universal central extension of $\mathcal{E}(R)$, and consequently that $\pi_2 B^+_{G\ell(R)} \cong K_2(R)$.

Definition 2.4. $K_n(R) = \pi_n B^+_{G\ell(R)}$, $n \geq 1$.

The space $B^+_{G\ell(R)}$ is an extraordinary space. Quillen proves that it is a homotopy commutative and associative H-space, and in fact is an infinite loop space. One of the most useful properties of $B^+_{G\ell(R)}$ is expressed in the following result.

Theorem 2.5. The map $B_{G\ell(R)} \xrightarrow{f} B^+_{G\ell(R)}$ is universal for maps $B_{G\ell(R)} \xrightarrow{g} H$ where H is an H-space. That is, there are maps $B^+_{G\ell(R)} \xrightarrow{h} H$ such that the diagram

$$\begin{array}{ccc} B_{G\ell(R)} & \xrightarrow{g} & H \\ \downarrow f & \nearrow h & \\ B^+_{G\ell(R)} & & \end{array}$$

is homotopy commutative, and such that the induced map $[X, B^+_{G\ell(R)}] \xrightarrow{h_*} [X,H]$ is independent of choice of h for all finite CW complexes X.*)

The proof is an application of obstruction theory. For example, one notes that the obstructions to constructing h lie in $H^*(B^+_{G\ell(R)}, B_{G\ell(R)}; \pi_* H)$. Since H is n-simple for all n, the

*) If X,Y are spaces [X,Y] denotes the set of homotopy classes of maps $X \longrightarrow Y$.

kernel $\pi_1 f = \mathcal{E}(R)$ is contained in kernel $\pi_1(g)$, and consequently the local coefficient system is trivial. These relative groups are then zero, since the fibre F of f is acyclic.

Suppose now that G is a group and $\mathcal{P}_A(G)$ is the category of representations

$$G \xrightarrow{\rho} \mathrm{Aut}_A P$$

of G in finitely generated projective A-modules. Let $R_A(G)$ denote the Grothendieck group of the category $\mathcal{P}_A(G)$ with relations all short exact sequences of representations. $R_A(G)$ is a contravariant functor of G and covariant in A. Let X be a finite based CW complex.

<u>Theorem 2.6</u>. There is a morphism

$$R_A(\pi_1(X)) \xrightarrow{\eta_X} [X, K_0(A) \times B_{G\ell A}^{+}]$$

(natural in X) which is universal for morphisms $R_A(\pi_1(X)) \xrightarrow{\xi_X} [X,H]$, where H is an H-space. That is, there are maps $K_0(A) \times B_{G\ell A}^{+} \xrightarrow{\omega} H$ such that the diagram

$$\begin{array}{ccc} R_A(\pi_1(X)) & \xrightarrow{\xi_X} & [X,H] \\ \Big\downarrow \eta_X & \nearrow \omega_* & \\ [X, K_0(A) \times B_{G\ell(A)}^{+}] & & \end{array}$$

commutes and ω_* is independent of the choice of ω.

We indicate how to define η_X. If $\pi_1 X \xrightarrow{\rho} \mathrm{Aut}_A(P)$ is a representation of $\pi_1 X$, then by choosing a complement Q for P (so that $P \oplus Q \cong A^n$) we get

$$\pi_1 X \xrightarrow{\rho} \mathrm{Aut}_A(P) \longrightarrow \mathrm{Aut}_A(P\oplus Q) \cong G\ell_n(A) \longrightarrow G\ell(A).$$

This determines a map $B_{\pi_1 X} \longrightarrow B_{G\ell(A)}$. Composing with the 2-coskeleton $X \longrightarrow B_{\pi_1(X)}$ we have a map

$$X \longrightarrow B_{G\ell(A)} \longrightarrow B_{G\ell(A)}^{+} .$$

The component of the desired map of X into $K_0(A)$ is just the class of P. One must verify of course that this association is independent of choices and preserves the relations of $R_A(\pi_1(X))$. This is non-trivial and should appear in a forthcoming paper of Quillen [40].

For computations of K_i of various rings, we refer the reader to Quillen's talk in these proceedings. We mention some general results about the Quillen K-groups.

Theorem 2.7 [42]. if R is noetherian and left regular, then $K_i(R) = K_i(R[t])$ for all i. Equivalently,

$$H_i(G\ell(R), \mathbb{Z}) = H_i(G\ell(R[t]), \mathbb{Z}) \quad \text{, all i.}$$

If R is a left noetherian ring, then Choo [9] has shown that the free associative algebra $\Lambda = R\{X\}$ on the set X is a coherent ring.*) Similar arguments also show that if $g\ell \dim R \leq n$, then $g\ell \dim \Lambda \leq n+1$. The ideas used in the proof of 2.7 can then be used to prove

Theorem 2.8. If R is left noetherian ring of finite global dimension, then $K_i(R\{X\}) = K_i(R)$, all i, where $R\{X\}$ is the free associative algebra on the set X.

*) See also Choo, Lam, and Luft [59].

As for the Laurent polynomials, we have

Theorem 2.9. If R is any ring, then

$$K_n(R[t,t^{-1}] \cong K_n(R) \oplus K_{n-1}(R) \oplus (?)$$

The proof for n = 2 was given for Wagoner [55] and for general n by Gersten [18]. Quillen has recently proved that the term (?) is zero if R is noetherian and left regular [42]. The general structure of the term (?) is still a mystery but one suspects that it is related to the category Nil (R) of pairs (P,ν) consisting of a projective R module P and a nilpotent endomorphism ν of P (compare [56]).

We shall describe now a delooping of the space $B^+_{G\ell(R)}$ due to Wagoner [56] and Gersten [18]. Let E be the ring of infinite matrices $(r_{ij}), r_{ij} \in R;\ i,j \geq 1$, where all but finitely many entries in any row or column are zero. Let CR be the subring of E generated by (infinite) permutation matrices and diagonal matrices $d = \text{diag}\ (d_1, d_2, d_3, \ldots)$ of finite type, in the sense that the diagonal entries $\{d_i\}$ of d are selected from a finite subset of R.

Theorem 2.10. $B^+_{G\ell(CR)}$ is contractible and $K_0(CA) = 0$.

The fact that $K_0(CA) = K_1(CA) = 0$ was proved first by Karoubi [26], who observed that there is an endomorphism τ of the category $\mathcal{P}_{CR}$ such that

$$\tau \oplus 1 \cong \tau .$$

This fact was exploited in [18] and [56] to prove 2.10.

Next let $\widetilde{R} = \bigcup_{n\geq 0} M_n(R)$ be the two sided ideal of "finite" matrices in C(R), where $M_n(R) \hookrightarrow M_{n+1}(R)$ by $M \longmapsto \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}$.

One lets $SR = CR/\widetilde{R}$.

Theorem 2.11 [18],[56]. The fibre of the canonical map $B^+_{G\ell(CR)} \longrightarrow B^+_{G\ell(SR)}$ is homotopy equivalent to $K_0(R) \times B^+_{G\ell(R)}$. Consequently we have

2.12) $$\Omega B^+_{G\ell(SR)} \simeq K_0(R) \times B^+_{G\ell(R)}$$

and hence $K_{i+1}(SR) = K_i(R)$, $i \geq 0$.

There is a particularly interesting application of this result. For we construct an Ω-spectrum $E(R)$ [18] whose n^{th} space is

$$E(R)_n = K_0(S^nR) \times B^+_{G\ell(S^n(R))}; \quad n \geq 0$$

and canonical homotopy equivalence

$$E(R)_n \simeq \Omega E(R)_{n+1}$$

is given by 2.12). As usual, one sets $\pi_i(E(R)) = \lim\limits_{n\to\infty} \pi_{i+n}(E(R)_n)$, where one observes that the definition $\pi_i(E(R))$ makes sense for all $i \in \mathbb{Z}$, and in fact $\pi_i(E(R)) = K_{i+n}(S^nR)$, $n+i \geq 0$, $n \geq 0$. However the groups $K_0(S^nR) = \pi_{-n}(E(R))$ have been identified by Karoubi [25] as $(L^nK_0)(R)$, Bass' candidate for $K_{-n}(R)$, $n \geq 0$ [3]. We recall that if F: Ring $\longrightarrow$ Ab is a functor, then LF: Ring $\longrightarrow$ Ab is defined by exactness in the diagram

$$F(R[t]) \oplus F(R[t^{-1}]) \longrightarrow F(R[t,t^{-1}]) \longrightarrow (LF)(R) \longrightarrow 0 .$$

Theorem 2.13. The spectrum $E(R)$ has homotopy groups $\pi_i E(R) = K_i(R)$, $i \geq 0$ and $\pi_{\bar{i}} E(R) = L^iK_0(R) = K_0(S^iR)$, $i \geq 0$. If R is noetherian and left regular, then $L^iK_0(R) = 0$ for all $i > 0$.

To terminate this section we collect some alternative procedures for constructing the space $B^+_{G\ell(R)}$. The first, due to E. Dror, makes use of the integral completion functor $\mathbb{Z}_\infty$ of Kan and Bousfield [8]. The functor $\mathbb{Z}_\infty$ is defined on simplicial sets having one vertex and takes values in the same category. Its main properties can be summarized as follows.

2.14. If $f: X \longrightarrow Y$ induces an isomorphism on integral homology, then $\mathbb{Z}_\infty(f): \mathbb{Z}_\infty X \longrightarrow \mathbb{Z}_\infty Y$ is a weak homotopy equivalence.

2.15. There is a morphism $X \xrightarrow{i} \mathbb{Z}_\infty X$, which is a weak homotopy equivalence if X is an H-space (or more generally if X is nilpotent [8]).

Theorem 2.16 (E. Dror). $B^+_{G\ell(R)} \simeq \mathbb{Z}_\infty B_{G\ell(R)}$.

Proof: Consider the diagram

$$\begin{array}{ccc} B_{G\ell R} & \xrightarrow{\quad f \quad} & B^+_{G\ell(R)} \\ \downarrow & & \downarrow \\ Z_\infty B_{G\ell(R)} & \xrightarrow{\quad \mathbb{Z}_\infty(f) \quad} & Z_\infty B^+_{G\ell(R)} \end{array}$$

By 2.1, f induces an isomorphism on integral homology. By 2.14, $\mathbb{Z}_\infty(f)$ is a weak homotopy equivalence. But $B^+_{G\ell R}$ is an H-space, so $B^+_{G\ell(R)} \longrightarrow \mathbb{Z}_\infty B^+_{G\ell(R)}$ is a weak homotopy equivalence. This completes the proof.

Next, the Anderson-Segal approach [1],[46]. The category $\mathcal{P}_R$ with its product $\oplus$ is first replaced by an equivalent category with product $\mathcal{C}$, $\perp$ where the operation $\perp$ is strictly associative and satisfies a type of commutative law. The nerve $B_{\mathcal{C}}$ of the category $\mathcal{C}$ acquires from $\perp$ the structure of a simplicial monoid. But one may group complete this simplicial monoid by taking its classifying space $B(B_{\mathcal{C}})$ ([45]; the Bar construction) and then applying Kan's free loop group functor G [34] to get $GB(B_{\mathcal{C}})$.

Theorem 2.17. $GB(B_{\mathcal{C}}) \simeq K_0(R) \times B_{G\ell(R)}^{+}$.

A proof of 2.17 may be found in [39]. Anderson and Segal show how to produce a generalized cohomology theory from any category with product, coherently associative and commutative in the sense of MacLane [58]. They produce in particular deloopings of the space $GB(B_{\mathcal{C}})$. Their spectrum is connected, in the sense that $\pi_i = 0$ for $i < 0$. Thus, their delooping is different from that of 2.12) if R is not regular. The relationship between these deloopings is still unclear.

A result that follows from the Anderson-Segal approach is the following

Theorem 2.18. Suppose R and S are Morita equivalent rings (that is they have equivalent categories of left-modules) then $K_i(R) \cong K_i(S)$ for all i.

We have also avoided the question of extending the Mayer-Vietoris sequence 1.2 and 1.8. In fact, see Swan's talk on lack of excision to see that this cannot in fact be done. However we have the following

Theorem 2.19. Let R be a commutative noetherian regular ring and let $(f,g) = R$. Then there is an exact Mayer-Vietoris sequence

$$\cdots \longrightarrow K_{n+1}(R_{fg}) \longrightarrow K_n(R) \longrightarrow K_n(R_f)\oplus K_n(R_g) \longrightarrow K_n(R_{fg}) \longrightarrow \cdots$$

Here R_f is the localization of R at the monoid of powers of f.*)

The result 2.19 was first proved for the Karoubi-Villamayor theory (see §3 below) by Gersten [21]. It is now known that the Quillen theory and the Karoubi-Villamayor theory agree on regular rings (Proposition 3.14 below). A second direct proof follows from Quillen's theory of localizations [42].

We mention now some features of the Quillen theory. These are external pairings $K_i(A) \otimes K_j(B) \longrightarrow K_{i+j}(A \otimes_{\mathbb{Z}} B)$. If A is commutative, then the map $A \otimes_{\mathbb{Z}} A \longrightarrow A$ gives internal pairings $K_i(A) \otimes K_j(A) \longrightarrow K_{i+j}(A)$, which make $K_*(A)$ into an anti-commutative graded ring ($x\cdot y = (-1)^{ij} y\cdot x$ for $x \in K_i(A), y \in K_j(A)$). If $A \xrightarrow{f} B$ is a ring homomorphism and B has a finite resolution by finitely generated projective A-modules, then the transfer $K_i(B) \xrightarrow{f_*} K_i(A)$ is defined [42]. In addition, if $f^*: K_i(A) \longrightarrow K_i(B)$ is induced by $B\otimes_A: \mathcal{P}_A \longrightarrow \mathcal{P}_B$, then the projection formula is valid if A and B are commutative:

$$f_*(f^*(a)\cdot b) = a\cdot f_*(b) \quad , \quad a \in K_i(A),\ b \in K_j(B) \ .$$

Since $f^*: K_* A \longrightarrow K_* B$ is a ring homomorphism, the projection formula states that f_* is a homomorphism of $K_*(A)$-modules.

In the special case $f: A \longrightarrow B$ where B is projective and of finite type as left A-module, the transfer can be constructed quite directly. Given an arbitrary representation ρ of a group G in a projective B module P of finite type, $G \xrightarrow{\rho} \mathrm{Aut}_B P$, by

*) J. P. Jouanolou [61] has recently extended this result to a higher K-theory of schemes. The Mayer-Vietoris sequence for schemes also appears in the paper of K. Brown and S. M. Gersten [62].

restriction of operators, ρ determines a representation $A \xrightarrow{\rho_1} \mathrm{Aut}_A P$, where P is projective and of finite type as an A-module. The assignment $\rho \longmapsto \rho_1$ is an exact functor $\mathcal{P}_B(G) \longrightarrow \mathcal{P}_A(G)$, hence determines a natural transformation

$$R_B(G) \longrightarrow R_A(G)$$

of contravariant functors in G. If now X is a based finite CW complex, we have the morphisms η of Theorem 2.6, and the universal property of 2.6 determines a commutative diagram of functors in X:

$$\begin{array}{ccc} R_B(\pi_1(X)) & \longrightarrow & R_A(\pi_1(X)) \\ \eta \downarrow & & \downarrow \eta \\ [X, K_0(B) \times B^+_{G\ell(B)}] & \xrightarrow{\omega} & [X, K_0(A) \times B^+_{G\ell(A)}] \end{array}$$

Taking $X=S^n$, the induced map $K_n(B) \longrightarrow K_n(A)$ is the desired transfer.

We should like to discuss now a method of approximating the space $B^+_{G\ell(R)}$ by a Postnikov type tower. The construction is due to Dror in his thesis [10], where in fact it is done in much greater generality. It will be clear that the tower can (and should) be constructed as a functor in simplicial sets, but we adhere to the looser geometric language we've been using. Let $X_0 = B_{G\ell(R)}$, and observe that the maximal perfect subgroup of $G\ell(R)$ is $\mathcal{E}(R)$. Let X_1 be the covering space of X_0 corresponding to $\mathcal{E}(R)$, so we have the fibration

$$X_1 \longrightarrow X_0 \longrightarrow K(K_1(R),1)$$

and $X_1 \simeq B_{\mathcal{E}(R)}$. Observe now that $H_1(X_1) = 0$, so $H^2(X_1, H_2(X_1)) = \mathrm{Hom}(H_2(X_1), H_2(X_1))$ by universal coefficients. Representing the identity map, we construct the fibration

$$X_2 \longrightarrow X_1 \longrightarrow K(H_2(X_1),2).$$

The exact homotopy sequence of this fibration shows that $\pi_i(X_2) = 0$ for $i \geq 2$ and yields the central extension

$$0 \longrightarrow H_2(X_1) \longrightarrow \pi_1 X_2 \longrightarrow \pi_1 X_1 \longrightarrow 0.$$

The low degree terms of the Serre spectral sequence show that $H_1(X_2) = H_2(X_2) = 0$. Since $X_1 = B_{\mathcal{E}(R)}$, it follows that $H_2(X_1) = K_2(R)$ and that $X_2 \simeq B_{St(R)}$ (compare proof of 2.2).

Now $H^3(X_2, H_3(X_2)) = \mathrm{Hom}(H_3(X_2), H_3(X_2))$, so again, representing the identity map, we get a fibration

$$X_3 \longrightarrow X_2 \longrightarrow K(H_3(X_2),3).$$

The homotopy exact sequence shows that $\pi_1 X_3 = \pi_1 X_2 = St(R)$ and $\pi_i X_3 = 0$, for $i \geq 3$, whereas $\pi_2 X_3 = H_3 X_2$. In addition, the Serre spectral sequence shows that $H_i(X_3) = 0$ for $i \leq 3$.

This process of killing successive homology groups may be continued to construct fibrations

$$X_{n+1} \longrightarrow X_n \longrightarrow K(H_{n+1}(X_n), n+1) \quad .$$

We summarize the properties of the X_n.

Proposition 2.20.
$$\pi_i(X_n) = 0 \ , \quad i \geq n \geq 2 \ .$$
$$\pi_i(X_n) = H_{i+1}(X_i) \ , \quad 2 \leq i < n.$$
$$H_i(X_n) = 0 \ , \quad 1 \leq i \leq n \ .$$
$$\pi_1(X_n) = St(R) \ , \quad n \geq 2 \ .$$

Hence if we let X_∞ be the "limit" of the tower $\cdots \longrightarrow X_{n+1} \longrightarrow X_n \longrightarrow \cdots$, we have $H_*(X_\infty, \mathbb{Z}) = H_*(pt, \mathbb{Z})$, $\pi_1 X_\infty = St(R)$, $\pi_i X_\infty = H_{i+1} X_i$, $2 \le i < n$.

Theorem 2.21. X_∞ is the fibre of the map $B_{G\ell R} \longrightarrow B^+_{G\ell(R)}$. Thus $\pi_n(X_\infty) = \pi_{n+1} B^+_{G\ell(R)} = K_{n+1}(R)$, $n \ge 2$.

Corollary 2.22. $K_3(R) = H_3(St(R), \mathbb{Z})$.

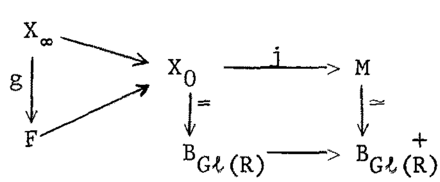
Proof of 2.22: By 2.21, $K_3(R) = \pi_2 X_\infty = H_3(X_2) = H_3(B_{St(R)})$. Another proof of 2.22 can be found in [20].

The following proof of 2.21 is due to Quillen. Set $M = X_0 \cup CX_\infty$, the mapping cone of the map $X_\infty \longrightarrow X_0 = B_{G\ell(R)}$, so one has the cofibration

$$X_\infty \longrightarrow X_0 \xrightarrow{j} M .$$

Now X_∞ is acyclic, so $X_0 \xrightarrow{j} M$ induces an isomorphism on integral homology. By Van Kampen's theorem, one sees that $\pi_1(j)$ is the map $G\ell(R) \longrightarrow\!\!\!\!\rightarrow K_1(R)$. By Quillen's theorem (see 2.1), the map j is identified with the map $B_{G\ell(R)} \longrightarrow B^+_{G\ell(R)}$. It suffices, therefore, to show that the cofibration $X_\infty \longrightarrow X_0 \xrightarrow{j} M$ is also a fibration.

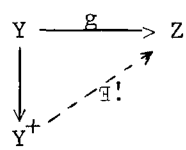
Let F be the fibre of $j: X_0 \longrightarrow M$. Since the composition $X_\infty \longrightarrow X_0 \longrightarrow M$ is null homotopic, there is a map $X_\infty \xrightarrow{g} F$ and a homotopy commutative diagram



It suffices to prove that g is a homotopy equivalence. We do this by constructing a homotopy inverse for g.

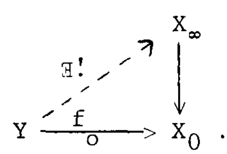
Lemma: If Y is acyclic and Z has abelian fundamental group, then $[Y,Z] = 0$.

Proof: Let $Y \longrightarrow Y^+$ be as in 2.1 with $E = \pi_1 Y$. Then by obstruction theory, there is a unique (up to homotopy) factorization

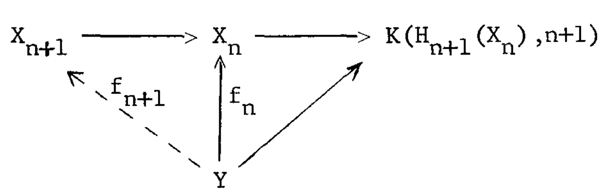


of any map $g: Y \longrightarrow Z$. But Y^+ is simply connected and acyclic, hence contractible. Thus $g \simeq 0$.

Corollary: If Y is acyclic, then there is a unique (up to homotopy) lift of any map $Y \xrightarrow{f_o} X_0$ to X_∞ :



Proof: Consider the problem of lifting a map $Y \xrightarrow{f_n} X_n$ to a map $Y \xrightarrow{f_n} X_{n+1}$. One has the fibration



and the spaces $K(H_n(X_n), n+1)$ all have abelian fundamental group. It follows one can lift f_n to a map f_{n+1}. The uniqueness statements follow similarly.

Apply the corollary to the map $F \longrightarrow X_0$ to construct a factorization $F \xrightarrow{h} X_\infty$:

$$\begin{array}{ccc} X_\infty & \longrightarrow & X_0 \\ g \downarrow \uparrow h & \nearrow & \\ F & & \end{array}$$

That $h \circ g \simeq 1$ follows from the uniqueness assertion of the corollary. In particular, g_* is injective on homotopy. The map $g \circ h$ is an endomorphism of the fibre in the fibration diagram

$$\begin{array}{ccccc} F & \longrightarrow & X_0 & \xrightarrow{j} & M \\ \downarrow g \circ h & & \downarrow = & & \downarrow = \\ F & \longrightarrow & X_0 & \xrightarrow{j} & M \end{array}$$

From the five lemma, it follows that g_* is surjective on homotopy. Thus, g is a homotopy equivalence and the proof of 2.21 is complete.

It is worth remarking that the tower of spaces $\cdots \longrightarrow X^+_{n+1} \longrightarrow X^+_n \longrightarrow \cdots \longrightarrow X^+_0$ is the usual "upside down" Postnikov tower of $X^+_0 = B^+_{G\ell(R)}$, where X^+_n kills the first n homotopy groups of X^+_0 [34].

§3. The Karoubi-Villamayor Theory.

The point of departure of the Karoubi-Villamayor Theory [27] is the functor K^h_1 introduced in §1. It is most convenient to set up this theory in the setting of rings without unit. Hence in this section $\mathcal{A}$ will denote the category of rings without unit and homomorphisms. Observe that each A in $\mathcal{A}$ determines a unitary ring $A^+ = \mathbb{Z} \times A$, where $(n,a)\cdot(m,b) = (nm, nb+ma+ab)$ and the addition is componentwise. The functor $A \longmapsto A^+$ is left adjoint to the forgetful functor <u>Ring</u> $\longrightarrow \mathcal{A}$.

If F: <u>Ring</u> $\longrightarrow$ <u>Group</u> is a product preserving functor, we may define F: $\mathcal{A} \longrightarrow$ <u>Group</u> by $F(A) = \mathrm{Ker}(F(A^+) \longrightarrow F(\mathbb{Z}))$, where $F(A^+) \longrightarrow F(\mathbb{Z})$ is induced by the augmentation. A check shows that this definition is consistent for unitary rings. Thus the functors $G\ell$, K_0, K_1^h, K_1, etc. are defined for rings without unit.

<u>Definition 3.1</u>. The <u>path ring</u> EA of A, A in $\mathcal{A}$, is the kernel of the map $A[t] \xrightarrow{t\to 0} A$. Thus EA consists of polynomials $\Sigma\, a_i t^i$ with zero constant term. There is a map $EA \xrightarrow{\varepsilon} A$ induced by $\Sigma\, a_i t^i \longrightarrow \Sigma\, a_i$. The <u>loop ring</u> ΩA is the kernel of ε: $EA \longrightarrow A$. Let μ: $EA \longrightarrow E^2A = t_1 t_2\, A[t_1, t_2]$ be given by $\sum_{i\geq 1} a_i t^i \longmapsto \sum_{i\geq 1} a_i (t_1 t_2)^i$. Then (E, ε, μ) is a cotriple in $\mathcal{A}$.

Thus one has an augmented simplicial ring

$$E^*A:\quad A \xleftarrow{\ \varepsilon\ } EA \rightleftarrows E^2A \;\substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow}\; E^3A \;\ldots$$

If we apply the functor $G\ell$ to E^*A, we get an augmented simplicial group $G\ell(E^*A)$. We set $K_{i+2}^h(A) = \hat{\pi}_i\, G\ell(E^*A)$, $i \geq -1$, where $\hat{\pi}_i = \pi_i$, $i \geq 1$, $\hat{\pi}_0(G\ell E^*A) = \mathrm{Ker}\ \pi_0(G\ell E^*A) \xrightarrow{\varepsilon} G\ell A$, and $\hat{\pi}_{-1}(G\ell E^*A) = \mathrm{Coker}\ (G\ell EA \longrightarrow G\ell A)$. This description was given in [16] and we shall find it convenient in comparing the K_i^h with the Swan groups K_i^S later.

The notation K_1^h is consistent, for one readily checks that if A is unitary, then the image $G\ell EA \longrightarrow G\ell A$ is exactly the subgroup UP(A) generated by unipotent matrices.

Lemma: $\qquad K_i^h(EA) = 0$, all $i \geq 1$.

For the augmented simplicial ring $E^*(EA)$ possesses then an extra degeneracy which gives the null homotopy of $G\ell(E^*A)$.

Observe that the split sequence

$$EA \longrightarrow A[t] \xrightarrow{t \to 0} A$$

gives rise to a split short exact sequence of simplicial groups

$$1 \longrightarrow G\ell(E^*EA) \longrightarrow G\ell(E^*A[t]) \longrightarrow G\ell(E^*A) \longrightarrow 1$$

and hence the homotopy exact sequences

$$0 \longrightarrow K_i^h(EA) \longrightarrow K_i^h(A[t] \longrightarrow K_i^h(A) \longrightarrow 0 .$$

Thus $K_i^h(A[t] \longrightarrow K_i^h(A)$ is an isomorphism and we conclude that K_i^h are homotopy functors $(i \geq 1)$ (see §1).

Definition 3.2. The homomorphism $B \xrightarrow{f} C$ is a $G\ell$ fibration if $G\ell(E^nB) \longrightarrow G\ell(E^nC)$ is surjective for all $n \geq 1$. For example, if C is unitary, noetherian, and left regular, then any surjection $f\colon B \twoheadrightarrow C$ is a $G\ell$ fibration.

Lemma 3.3. If $f\colon B \longrightarrow C$ is a $G\ell$ fibration and A is the kernel of f, then there is a long exact sequence

$$K_{n+1}^h(C) \longrightarrow K_n^h(A) \longrightarrow K_n^h(B) \longrightarrow K_n^h(C) \longrightarrow \cdots \longrightarrow K_1^h(C)$$

For combining the left exactness of $G\ell$ with the assumed right exactness in

$$1 \longrightarrow G\ell(E^nA) \longrightarrow G\ell(E^nB) \longrightarrow G\ell(E^nC) \longrightarrow 1 \quad (n \geq 0) ,$$

and taking the homotopy exact sequence yields the desired long exact sequence.

Suppose now that $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is an arbitrary short exact sequence of rings without unit. Then a result of [22] gives a natural six term exact sequence

3.4) $$K_1(A) \longrightarrow K_1(B) \longrightarrow K_1(C) \xrightarrow{\partial} K_0(A) \longrightarrow K_0(B) \longrightarrow K_0(C).$$

Proposition 3.5. [16,p. 75] If $g: B \longrightarrow C$ above is a $G\ell$ fibration, then 3.4) factors to give the exact sequence

$$K_1^h(A) \longrightarrow K_1^h(B) \longrightarrow K_1^h(C) \xrightarrow{\partial} K_0(A) \longrightarrow K_0(B) \longrightarrow K_0(C) .$$

Proof: $EA \longrightarrow EB \xrightarrow{Eg} EC$ is also a short exact sequence, so we have the commutative diagram with exact rows

$$\begin{array}{ccccc} K_1(EB) & \longrightarrow & K_1(EC) & \xrightarrow{\partial} & K_0(EA) \\ \downarrow & & \downarrow & & \downarrow \\ K_1(B) & \longrightarrow & K_1(C) & \longrightarrow & K_0(A) \end{array}$$

Since g is a $G\ell$ fibration, $G\ell EB \longrightarrow G\ell EC$ is surjective, hence $K_1EB \longrightarrow K_1EC$ is surjective. Thus the map $\partial: K_1EC \longrightarrow K_0EA$ is zero. It is now routine to verify the exactness of the resulting sequence of 3.5.

Lemma 3.6. For any A in $\mathcal{C}$, the map $EA \xrightarrow{\varepsilon} A$ is a $G\ell$ fibration.

Proof. It suffices to show $E(EA) \longrightarrow E(A)$ has a section. This map is

$$A \underset{\mathbf{Z}}{\otimes} tuZ[t,u] \xrightarrow{u \longmapsto 1} A \underset{\mathbf{Z}}{\otimes} tZ[t]$$

given by $(u \longmapsto 1)$. A section is defined by sending $t \longrightarrow tu$.

Corollary 3.7. $K^h_{n+1}(A) \cong K^h_n(\Omega A)$, $n \geq 1$.

For one writes down the exact K^h sequence of the short exact sequence $\Omega A \longrightarrow EA \xrightarrow{\varepsilon} A$.

Hence $K^h_{n+1}(A) \cong K^h_1(\Omega^n A)$, $n \geq 0$. To deal with K_0, we introduce the following

Definition 3.8*) The ring A is K-regular if for any set X, $K_0(A) \longrightarrow K_0(A[X])$ is an isomorphism. For example, any unital, regular, left noetherian ring is K-regular by Theorem 1.3. Karoubi [25] proves that if A is K-regular, then so are EA, ΩA, CA, and SA.

Lemma 3.9. If A is K-regular, then $K^h_1(A) \cong K_0(\Omega A)$ and hence also $K^h_n(A) \cong K_0(\Omega^n A)$.

For, consider the exact sequence of 3.5

$$K^h_1(EA) \longrightarrow K^h_1(A) \xrightarrow{\partial} K_0(\Omega A) \longrightarrow K_0(EA).$$

Observe that $K^h_1(EA) = 0$ and $K_0(EA) = 0$, by virtue of the short exact sequence

$$0 \longrightarrow K_0(EA) \longrightarrow K_0(A[t]) \xrightarrow{\cong} K_0(A) \longrightarrow 0 .$$

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*) This differs from Karoubi's terminology [25].

As for the Mayer-Vietoris sequence, we quote

Proposition 3.10 ([16] 2.10). Given a cartesian diagram in $\mathcal{A}$

$$\begin{array}{ccc} D & \longrightarrow & A \\ \big\downarrow & & \big\downarrow f \\ B & \xrightarrow{\ g\ } & C \end{array}$$

where g is surjective and a $G\ell$ fibration, then one has the long Mayer-Vietoris sequence

$$K_{n+1}^h(C) \longrightarrow K_n^h(D) \longrightarrow K_n^h(A) \oplus K_n^h(B) \longrightarrow K_n^h(C) \longrightarrow \cdots$$

$$\longrightarrow K_1^h(C) \longrightarrow K_0(D) \longrightarrow K_0(A) \oplus K_0(B) \longrightarrow K_0(C)$$

We have already mentioned the Mayer-Vietoris sequence of localization, for regular rings (2.19). Thus, the Karoubi-Villamayor theory has good exact sequences, for $G\ell$-fibrations, at the expense of the "right" K_1.

The Karoubi-Villamayor theory is equipped with products [28] and a transfer map for f: A ——> B where f is unitary and B is finitely generated and projective qua A module. Furthermore, this theory can be axiomatized [27].

One says a "K^h" theory is a sequence of functors $\varkappa_n : \mathcal{A} \longrightarrow \underline{\mathcal{Ab}}$, $n \geq 0$, such that for every surjective $G\ell$ fibration B ——> C with kernel A, one has a natural connecting homomorphism

$$\partial : \varkappa_{n+1}(C) \longrightarrow \varkappa_n(A)$$

and a long exact sequence

$$\varkappa_{n+1}(C) \xrightarrow{\ \partial\ } \varkappa_n(A) \longrightarrow \varkappa_n(C) \longrightarrow \cdots \longrightarrow \varkappa_0(C).$$

Theorem 3.11. There is one and only one K^h theory $\{\varkappa_n, n\geq 0\}$ up to isomorphism satisfying

(1) $\varkappa_n(A) = 0$, $n > 0$, if A is contractible (that is, if the identity map of A and 0 are homotopic maps $A \longrightarrow A$) and

(2) $\varkappa_0(A) = K_0(A)$.

To relate the Quillen theory K_i to that of Karoubi and Villamayor, we have

Theorem 3.12 If A is a unitary ring, there is a first quadrant spectral sequence of homological type, whose $E^1_{pq} = K_q(A[x_1,\dots,x_p])$, $q > 0$, converging to $K^h_{p+q}(A)$.

The argument can be found in [19]. It depends on an alternative description of the groups K^h_i due to D. Rector

Theorem 3.13 [43] Let R_* be the simplicial ring $R_n = R[t_0,\dots,t_n]/(t_0+\dots+t_n-1)$, where faces $R_n \xrightarrow{d_i} R_{n-1}$ are given by $t_i \longrightarrow 0$. Then $\pi_i G\ell(R_*) = K^h_{i+1}(R)$, $i \geq 0$.

A useful consequence of 3.12 is

Proposition 3.14. Suppose that, for all q, $K_q(R) \longrightarrow K_q(R[X])$ is an isomorphism for every set X. Then the edge homomorphism induces an isomorphism $K_q(R) \xrightarrow{\cong} K^h_q(R)$.

Quillen has recently shown that the hypotheses of 3.14 are satisfied if R is unitary, left noetherian, and left regular [42]. Thus the K_q and K^h_q theories agree on such rings. Furthermore, if R is unitary, of finite global dimension, and if R[X] is coherent for every set X, then again the hypotheses of 3.14 are satisfied. Hence $K_q(R) = K^h_q(R)$ for such rings.

There is also a delooping theorem [19].

Theorem 3.15. If R is unitary and K-regular, then

$$\Omega B_{G\ell(SR)_*} \simeq K_0(R) \times B_{G\ell(R)_*} .$$

Here $B_{G\ell(R)_*}$ is the simplicial classifying space for the simplicial group $G\ell(R_*)$ of 3.13.

As for Laurent polynomials, we quote

Theorem 3.16 ([25],[17]). If R is any K-regular ring then $K^h_{n+1}(R[t,t^{-1}]) \cong K^h_{n+1}(R) \oplus K^h_n(R)$ for $n \geq 0$. (denote $K^h_0 = K_0$).

For any ring R, let Ω_R (as distinct from ΩR) be the subring of R[t] consisting of polynomials $\sum_{i\geq 0} r_i t^i$ with $r_0 = \sum_{i\geq 0} r_i$.

Then there is a split short exact sequence of rings

$$\Omega R \longrightarrow \Omega_R \longrightarrow R$$

and consequently $K^h_i(\Omega_R) = K^h_i(\Omega R) \oplus K^h_i(R)$, $i \geq 0$, where we set $K^h_0 = K_0$. Let Ω^n_R be defined inductively by $\Omega^1_R = \Omega_R$, and $\Omega^{n+1}_R = \Omega_{(\Omega^n_R)}$ [36]. Then, if R is K-regular, we have by induction

$$K_0(\Omega^n_R) = \bigoplus_{i=0}^{n} \binom{n}{i} K^h_i(R) .$$

In particular, for a field F, one has

$$K_0(\Omega^2_F) = K_2(F) \oplus 2K_1(F) \oplus K_0(F) ,$$

where we have used the identification $K_i(F) = K^h_i(F)$. Now a presentation for $K_2(F)$ is known by work of Matsumoto [33]. One might hope that the formula above would be useful in providing an alternative proof of Matsumoto's theorem, by considering the geometry of the ring Ω_F^2. There also exists an elementary proof for the localization exact sequence of a dedekind k-algebra A (k a field) [23], which makes use of the fact that $K_n(A)$ can be recovered from $K_0(\Omega_A^n)$. Now that the theories K_i^h and K_i are known to agree on regular rings R, one might hope that one might be able to use geometric methods on (Ω_R^n) more systematically to gain information about $K_n(R)$.

Anderson [1] has another approach to the Karoubi-Villamayor theory. From the simplicial ring R_* (see 3.3) he constructs the simplicial category $\mathcal{P}_{R_*}$, $(\mathcal{P}_{R_*})_n = \mathcal{P}_{R_n}$, with the obvious faces and degeneracies. Then $B_{\mathcal{P}_*}$ is a bi-simplicial set, and if all the $\mathcal{P}_{R_n}$ are blown up to make them strictly associative under $\oplus$, then the diagonal complex of $B_{\mathcal{P}_{R_*}}$ becomes a simplicial monoid M. The group completion $\hat{M}$ of M is then homotopy equivalent to $B_{G\ell(R_*)}$.

The advantage of this approach is that it shows the dependence of K_i^h on the categories of projective modules on the rings R_n. If, for example, R and S are unitary rings and Morita equivalent, then $\mathcal{P}_{R_n}$ is equivalent to $\mathcal{P}_{S_n}$, all n, so a spectral sequence argument shows

<u>Proposition 3.17</u>. If R and S are unitary Morita equivalent rings, then

$$K_i^h(R) \cong K_i^h(S) \quad \text{for all } i \geq 0.$$

§4. The Theory of Gersten and Swan.

Swan introduced higher K-functors in [50] by resolving the functor $G\ell$ in the category of functors, and Gersten introduced higher K-functors [15] by introducing a cotriple construction in the category of rings and applying this to the elementary group $\mathcal{E}$. Swan later showed [52] that Gersten's resolution applied to $G\ell$ gave Swan's groups. This theory we shall denote by K_i^S, $i \geq 0$.

As in §3, $\mathcal{A}$ is the category of rings R without unit. The free ring (without unit) on a set S is the functor Set ⟶ $\mathcal{A}$ adjoint to the forgetful functor. Denote by FR the free ring on the underlying set of R. Then adjointness gives a morphism ϵ: FR ⟶ R and a morphism μ: FR ⟶ F^2R, so that (F, ϵ, μ) is a cotriple in $\mathcal{A}$. We denote by $|r|$ the free generator of FR corresponding to $r \in R$. Thus, $\epsilon(|r|) = r$, and $\mu(|r|) = \|r\|$. Thus one has the augmented simplicial ring F^*R

$$F^*R: R \longleftarrow FR \;\leftleftarrows\; F^2R \;\;\cdots\;\; F^3R \ldots$$

<u>Definition 4.1</u>. $K_i^S(R) = \hat{\pi}_{i-2}\, G\ell F^*(R)$, $i \geq 1$, where $\hat{\pi}_i$ has the same meaning as in 3.1. Set $K_0^S = K_0$.

<u>Theorem 4.2</u>. $K_1^S(R) = K_1(R)$.

The proof is based on Gersten's theorem, 1.4 above.

<u>Lemma</u>. $K_i^S(FR) = 0$, all $i \geq 0$.

For the simplicial ring F^*FR has an extra degeneracy. The result $K_0(FR) = 0$ was first proved by Bass [5].

There is an immediate relationship between the functors K_i^S and K_i^h. For there is a morphism of cotriples $(F,\varepsilon,\mu) \longrightarrow (E,\varepsilon,\mu)$ given by the homomorphism $FR \longrightarrow ER$, $|r| \longmapsto rt$, which induces maps $K_i^S(R) \longrightarrow K_i^h(R)$. For $i = 1$, this is just reduction modulo unipotents.

It has proved more difficult to relate the functors K_i to K_i^S. Presumably there is a spectral sequence, whose edge homomorphism is the desired map $K_i \longrightarrow K_i^S$, but this is still conjectural. However, the relation of K_2 and K_2^S is easily described.

Let R be a unitary ring and let G(R) be the free group on symbols $X_{ij}(r)$, $1 \leq i \neq j$; $r \in R$. There is a natural map $G(R) \longrightarrow St(R)$ given by

$$X_{ij}(R) \longmapsto x_{ij}(r)$$

Define a map $G(R) \longrightarrow G\ell\,(FR)$ by $X_{ij}(r) \longmapsto E_{ij}(|r|)$. Recall that $\pi_0 G\ell(F^*R)$ is the difference cokernel (coequalizer) of maps $G\ell\ F^2R \underset{\varepsilon F}{\overset{F\varepsilon}{\rightrightarrows}} G\ell\ FR$.

<u>Proposition 4.3</u> ([15],[52]) The composition

$$G(R) \longrightarrow G\ell(FR) \longrightarrow \pi_0 G\ell\ F^*R$$

factors through the map $G(R) \longrightarrow St(R)$.

Consider now the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
1 & \longrightarrow & K_2(R) & \longrightarrow & St(R) & \longrightarrow & \mathcal{E}(R) & \longrightarrow & 1 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \check{\pi}_0 G\ell F^*R & \longrightarrow & \pi_0 G\ell F^*R & \xrightarrow{\ \varepsilon\ } & G\ell(R) & &
\end{array}$$

There is an induced map

$$K_2(R) \xrightarrow{\omega} K_2^S(R) = \hat{\pi}_0 G\ell(F^*R).$$

It is easy to see that this map is surjective, and one has a commutative diagram

$$\begin{array}{ccc} K_2R & \xrightarrow{\omega} & K_2^S(R) \\ & \searrow \quad \swarrow & \\ & K_2^h(R) & \end{array}$$

where the map $K_2(R) \longrightarrow K_2^h(R)$ is the edge homomorphism of the spectral sequence 3.12. In [50] (theorem 8.5) it is shown that the kernel of ω: $K_2(R) \longrightarrow K_2^S(R)$ is generated by all images $K_2(FS) \longrightarrow K_2(R)$ under all ring homomorphisms $FS \longrightarrow R$, where FS is a free ring (without unit). (One interprets $K_2(A) = \text{Ker } K_2(A^+) \longrightarrow K_2(\mathbf{Z})$ for A in $\mathcal{Q}$). However $K_i(FS) = 0$ for all i, by Theorem 2.8. Hence, we have

<u>Theorem 4.4</u>. For any ring R, the natural map $K_2(R) \xrightarrow{\omega} K_2^S(R)$ is an isomorphism.

We have just seen this for unitary R. If R is without unit, consider the diagram (using the fact that K_2 preserves products)

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_2(R) & \longrightarrow & K_2(R^+) & \longrightarrow & K_2(\mathbf{Z}) \\ & & \downarrow & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & K_2^S(R) & \longrightarrow & K_2^S(R^+) & \longrightarrow & K_2^S(\mathbf{Z}) \end{array}$$

We might inquire whether a similar result holds for K_i^S, but this appears to demand construction of a spectral sequence.

An alternative method for constructing the K_i^S has recently been proposed by Keune [31]. If X is a set, let FX be the free ring without unit on X. The augmented simplicial ring A_* (without unit) is called free if A_i is free on X_i, $i \geq 0$, and if the degeneracies $s_j\colon A_i \to A_{i+1}$ preserve free generators: $s_j(X_i) \subset X_{i+1}$.

Let the n-cycles $Z_n(A_*)$ be

$$\{(a_0,\ldots,a_{n+1}) \in (A_n)^{n+2} \mid d_i a_j = d_{j-1}a_i \ , \ i<j\} \ .$$

Define d: $A_{n+1} \to Z_n(A_*)$ by $d(a) = (d_0 a,\ldots,d_{n+1}a)$. Then A_* is aspherical if d is surjective, all $n \geq -1$; A_* is a resolution of A_{-1} if it is aspherical; and A_* is a free resolution of A_{-1} if it is free and aspherical. One shows that any ring A_{-1} without unit has a free aspherical resolution (for example, F^*A_{-1} will work).

Proposition 4.5 [31]. Let A_*, B_* be augmented simplicial rings and $\varphi_{-1}\colon A_{-1} \to B_{-1}$ a homomorphism. Let A_* be free and let B_* be aspherical. Then there exists a unique (up to simplicial homotopy) simplicial homomorphism $\varphi\colon A_* \to B_*$ which extends φ_{-1}.

One proceeds now in the standard way. If R is a ring, let R_* be any free resolution of R.

Corollary 4.6. $\quad \hat{\pi}_{n-2}G\ell(R_*) = K_n^S(R) \ , \quad n \geq 1.$

Keune's method is particularly attractive since it removes one's dependence on the particular resolution F^*R, and is very suggestive of the approach taken in ordinary abelian homological algebra to define derived functors. If one could establish the identity of K_i^S with K_i, then K_i could be interpreted, in a very real sense, as derived functors of K_0

§5. Additional Remarks on K_1.

There is considerable activity now in extending the higher K-theory of rings to schemes [24],[42] and to abelian categories [42]. Since these ideas are presented elsewhere in this volume, we want to point out here that although K_0 is well understood (at any rate, one agrees how to define it) the same cannot be said about K_1. In this volume Quillen presents a construction*) which assigns to an abelian category $\mathcal{A}$ a new category $Q(\mathcal{A})$ whose objects are those of $\mathcal{A}$ and whose morphisms $A \longrightarrow A'$ are isomorphisms of A with a quotient of a subobject of A'. He defines $K_i(\mathcal{A}) = \pi_{i+1}(B_{Q(\mathcal{A})})$, the homotopy of the geometrical realization of the nerve of $Q(\mathcal{A})$, and verifies that $K_0(\mathcal{A})$ is the same group defined in §1. These groups are the "right" K-theory of $\mathcal{A}$, in view of Quillen's localization theorem.

However, Bass [3] defined a K_1 for an abelian category, which we denote here $K_1^{det}(\mathcal{A})$. One constructs the category $\Omega\mathcal{A}$ of pairs (A,α), A in $\mathcal{A}$ and α an automorphism of A, where morphisms are commutative diagrams. $\Omega\mathcal{A}$ has a notion of short exact sequence. These form a class $\mathcal{E}$ and one sets

$$K_1^{det}(\mathcal{A}) = K_0(\Omega\mathcal{A},\mathcal{E})/R$$

*) See Quillen's talk for precise details [42].

where R is the subgroup generated by

$$(A,\alpha) + (A,\beta) - (A,\alpha\beta).$$

There is a functor $\Omega\mathcal{G} \longrightarrow \mathcal{G}^{\mathbf{Z}}$ and this induces a morphism

$$Q(\Omega\mathcal{G}) \longrightarrow Q(\mathcal{G}^{\mathbf{Z}}) \to Q(\mathcal{G})^{\mathbf{Z}} .$$

Taking classifying spaces, gives

$$B_{Q(\Omega\mathcal{G})} \longrightarrow \underline{\mathrm{Map}}(B_{\mathbf{Z}}, B_{Q(\mathcal{G})}) \longrightarrow \Omega B_{Q(\mathcal{G})} \cdot$$

and applying π_1 gives

$$K_0(\Omega\mathcal{G},\mathcal{E}) \longrightarrow K_1(\mathcal{G}) .$$

One checks that this map kills the subgroup R, and hence there is a natural map

$$K_1^{det}(\mathcal{G}) \longrightarrow K_1(\mathcal{G}) .$$

In general, this map is neither injective (Gersten) nor surjective (Murthy). This leaves open the problem of studying the nature of $K_1(\mathcal{G})$. We shall sketch here the arguments of Murthy and Gersten.

If A is a noetherian ring, let $\mathfrak{m}_A$ be the category of finite A-modules, and $G_i(A) = K_i(\mathfrak{m}_A)$. Let $G_1^{det}(A) = K_1^{det}(\mathfrak{m}_A)$.

Proposition 5.1 (Murthy). If π is cyclic of order 2, then $G_1^{det}(\mathbf{Z}\pi) \longrightarrow G_1(\mathbf{Z}\pi)$ is not surjective.

Proof: Quillen has established an exact sequence of localizations [42] for abelian categories which, for the localization $\mathfrak{m}_{\mathbf{Z}\pi} \longrightarrow \mathfrak{m}_{\mathbf{Z}'\pi}$ (where $\mathbf{Z}' = Z[\frac{1}{2}]$), reads

$$\longrightarrow G_1(\mathbb{F}_2\pi) \longrightarrow G_1(\mathbb{Z}\pi) \longrightarrow G_1(\mathbb{Z}'\pi) \longrightarrow G_0(\mathbb{F}_2\pi) \longrightarrow G_0(\mathbb{Z}\pi) \longrightarrow G_0(\mathbb{Z}'\pi)$$

We claim rank $G_1(\mathbb{Z}\pi) \geq 1$. To see this, note that $G_0(\mathbb{F}_2\pi) \cong \mathbb{Z}$ since $\mathbb{F}_2\pi$ has one simple module. Similarly, $G_1(\mathbb{F}_2\pi) = G_1(\mathbb{F}_2\pi/\mathfrak{n}) = G_1(\mathbb{F}_2) = 0$, where $\mathfrak{n}$ is the nilradical of $\mathbb{F}_2\pi$. Since $\mathbb{Z}'\pi = \mathbb{Z}' \times \mathbb{Z}'$, it follows easily that $G_1(\mathbb{Z}'\pi)$ is of rank 2. Thus, the localization sequence becomes

$$0 \longrightarrow G_1(\mathbb{Z}\pi) \longrightarrow \underset{\text{rank } 2}{G_1(\mathbb{Z}'\pi)} \longrightarrow \underset{\text{rank } 1}{G_0(\mathbb{F}_2\pi)}$$

and we get rank $G_1(\mathbb{Z}\pi) \geq 1$ as claimed.

However, $G_1^{det}(\mathbb{Z}\pi)$ has been computed in T. Y. Lam's thesis [32, Theorem 4.1]. The result is $G_1^{det}(\mathbb{Z}\pi) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Hence $G_1^{det}(\mathbb{Z}\pi) \longrightarrow G_1(\mathbb{Z}\pi)$ is not surjective.

<u>Proposition 5.2</u> (Gersten). If X is a complete nonsingular curve over the complex numbers of genus 1, then $K_1^{det}(X) \longrightarrow K_1(X)$ is not injective, where $K_1(X) = K_1$ (coherent $\mathcal{O}_X$-modules) and similarly for $K_1^{det}(X)$.

<u>Proof</u>: Let F be the field of rational functions of X, let D be the divisor group. Then the exact sequence of localization at the generic point is

5.3) $$\longrightarrow K_2(F) \xrightarrow{\partial} D\otimes\mathbb{C}^* \longrightarrow K_1X \longrightarrow F^* \xrightarrow{div} D \longrightarrow \text{Pic}X \longrightarrow 0.$$

Since $\mathbb{C}^*$ is the kernel of the divisor map, this gives

$$K_2F \xrightarrow{\partial} D\otimes\mathbb{C}^* \longrightarrow K_1X \longrightarrow \mathbb{C}^* \longrightarrow 0 .$$

But $K_1^{det}(X) \cong K_0(X)\otimes\mathbb{C}^*$ by Robert's theorem [3] and $K_0(X) = \text{Pic}X\oplus\mathbb{Z}$, so $K_1^{det}(X) = (\mathrm{Pic}X\otimes\mathbb{C}^*) \oplus \mathbb{C}^*$. It is easy to see that the following diagram is commutative

$$\begin{array}{ccccccccc}
 & & & & \mathrm{Pic}\ X\otimes\mathbb{C}^* & & & & \\
 & & & \nearrow & \downarrow & & & & \\
 & & & & K_1^{det}X & & & & \\
 & & & & \downarrow & & & & \\
K_2(F) & \longrightarrow & D\otimes\mathbb{C}^* & \longrightarrow & K_1X & \longrightarrow & \mathbb{C}^* & \longrightarrow & 0
\end{array}$$

where the map $D\otimes\mathbb{C}^* \twoheadrightarrow \mathrm{Pic}X\otimes\mathbb{C}^*$ is the natural projection. Thus, to show that $K_1^{det}(X) \longrightarrow K_1(X)$ is not injective, it suffices to prove that the composition

$$K_2F \xrightarrow{\partial} D\otimes\mathbb{C}^* \longrightarrow \mathrm{Pic}X\otimes\mathbb{C}^*$$

is not zero. However, the map ∂ is just the tame symbol; if $f,g \in F^*$, then

$$\partial\{f,g\} = \sum_{\substack{P\in X\\ \text{closed}}} P \otimes h_P(P) \quad ,$$

where $h_P = (-1)^{v_P(f)v_P(g)} \cdot \dfrac{f^{v_P(g)}}{g^{v_P(f)}}$.

Thus, it suffices to prove that some expression $\partial\{f,g\} \in D\otimes\mathbb{C}^*$ does not represent zero in $\mathrm{Pic}X\otimes\mathbb{C}^*$. However, this is in effect what L. Roberts showed in his thesis [44] (argument preceding Theorem 4.4.4) for X an elliptic curve.

It is perhaps worth pointing out that the exact sequence 5.3) gives a presentation for $SK_1(X) = \mathrm{Ker}(K_1X \longrightarrow \mathbb{C}^*)$, since the groups K_2F and $D\otimes C^*$ and the map ∂ are "known."

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