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HOMOLOGICAL ALGEBRA IN ABELIAN CATEGORIES

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Introduction

Homological algebra, as a connected system of notions and results, was first developed for categories of modules by H. Cartan and S. Eilenberg [2] and was immediately generalized by D. A. Buchsbaum to exact categories [1 and 2, appendix]. The generalization was an important one, since it subsumed such new applications as categories of sheaves and of F–D complexes, as well as allowing a completely self-dual development of the subject.

This generalization is not however sufficient to cover all the potential, or even all the actual, applications of homological algebra. This first became apparent to the author in connection with studies in the homotopy classification of chain-complexes of modules (to appear elsewhere). However, such categories as that of Banach spaces and continuous linear maps, or that of abelian varieties over a field of nonzero characteristic, ought clearly to have homological algebra, yet are not exact.

This paper, then, is yet another exposition of some of the notions of homological algebra, generalized from exact categories to a rather more inclusive context, that of abelian categories (§ 3). This is broad enough to contain most of the examples known to the author, and has also the advantage that in it the relative theory [3] is no longer distinguished from the absolute one. But new applications appear to be accumulating at such a rate that it seems futile to hope for a definitive treatment at this time.

The notion of an abelian category is grounded on that of an additive category (§ 1), i. e., a category in which maps can be added. In such a category exactness can be defined, as well as the notions of kernel, image and cokernel. It is not in general true however that every map can be asserted to have such appurtenances. If this is indeed the case (in a sufficiently precise way) then the category is an exact category in the sense of Buchsbaum. More generally an appropriate class of such maps, introduced as additional structure on the category, makes the category into an abelian category.

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Much of this paper is devoted to showing that the machinery of homological algebra operates in abelian categories. The novelties introduced in this procedure are mainly technical in character. For example, in proving exactness it is necessary to show that the maps in question do in fact have kernels, cokernels and images. It is perhaps remarkable that objects-with-derivation do not in general have homology, but always have cohomology. They have in fact two kinds of cohomology, dual in the categorical sense to each other and not, of course, to homology (§11).

An important innovation is the systematic use of categories whose objects are short exact sequences of an abelian category. These were unavailable in previous treatments since, though they are abelian, they are never exact (§ 6). In spite of this defect they are not at all unmanageable ; this is illustrated by the explicit computation for Ext in such a category, given in § 14.

The notions of category and functor are here taken for granted. It is supposed that a category is an honest mathematical object, so that such operations may be made as the formation of classes of subsets or the assignment of choice-functions. Of course this means that such familiar locutions as "the category of abelian groups" must be taken with the customary grain of salt. The notation is that of concrete categories, i. e., an object A of a category is distinguished from its identity map $1 : A$. The theory is completely self-dual with respect to the usual contravariant isomorphism between a category and its dual. The duals of definitions and theorems are usually stated ; the latter are never explicitly proved.

1. Additive categories

A *preadditive category* $\mathscr{K}$ is a category in which for each $A, B \in \mathscr{K}$ the set $\mathrm{Hom}(A, B ; \mathscr{K})$ – often abbreviated $\mathrm{Hom}(A, B)$ – of maps from A to B is provided with the structure of an abelian group in such a way that the following axiom holds.

(A) The composition of maps is a bilinear operation, i. e., for A, B, $C \in \mathscr{K}$ it is a homomorphism

$$\mathrm{Hom}(B, C) \otimes \mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(A, C) .$$

An object A of a preadditive category is a *zero-object* if $1 : A$ is the zero-element of $\mathrm{Hom}(A, A)$. Equivalently, A is a zero-object if $\mathrm{Hom}(A, A) = 0$, or if for all B, $\mathrm{Hom}(A, B) = 0$ and $\mathrm{Hom}(B, A) = 0$. Any two zero-objects are of course equivalent. The symbol "0" will be used to stand for a zero-object of any preadditive category.

If in a preadditive category, maps $i : A \to B$, $i' : A' \to B$, $j : B \to A$, $j' : B \to A'$ satisfy the relations

$$\begin{array}{ll} ji = 1 : A & ji' = 0 \\ j'i = 0 & j'i' = 1 : A' \\ \multicolumn{2}{c}{ij + i'j' = 1 : B} \end{array}$$

B is said to be a *direct sum* of A and A' relative to the *injections* i and i' and the *projections* j and j'. Alternatively, the situation may be described by saying that (A, A', i, j, i', j') is a *direct sum decomposition* of B.

PROPOSITION 1.1. *If* (A, A', i, j, i', j') *and* $(A, A', \bar{i}, \bar{j}, \bar{i}', \bar{j}')$ *are direct sum decompositions of* B *and* $\bar{B}$ *then* $\bar{i}j + \bar{i}'j' : B \to \bar{B}$ *and* $i\bar{j} + i'\bar{j}' : \bar{B} \to B$ *are inverse maps and thus equivalences. If* (A, A', i, j, i', j') *and* $(A, \bar{A}, i, j, \bar{i}, \bar{j})$ *are both direct sum decompositions of* B *then* $\bar{j}i' : A' \to \bar{A}$ *and* $j'\bar{i} : \bar{A} \to A'$ *are inverse maps and thus equivalences.*

For the last statement observe that

$$\bar{j}i'\, j'\, \bar{i} = \bar{j}(1 : B - ij)\bar{i} = \bar{j}\bar{i} = 1 : \bar{A}\,.$$

This proposition asserts that the summands determine the sum, and that the sum and one summand, together with its injection and projection, determine the other.

If a preadditive category $\mathscr{K}$ satisfies the two following conditions, it is an *additive category*.

(A0) $\mathscr{K}$ has a zero-object.

(A1) (Direct sum axiom). Any two objects of $\mathscr{K}$ have a direct sum.

While it is easy enough to construct preadditive categories which fail to satisfy either or both of (A01), such examples all seem sufficiently artificial to suggest that the notion of preadditive category can for the most part be by-passed. The categories considered below will all be additive categories.

Examples of additive categories may be considered under several headings.

I. Exact categories (Buchsbaum [1]) : the category of left modules over a given ring ; the category of sheaves of modules over a fixed sheaf of rings ; the category of F–D complexes, etc.

II. Subcategories of exact categories : in any of the above, a subclass of the objects, containing 0 and closed under direct sum, together with the maps connecting them ; in particular, the projectives or the injectives and the maps connecting them.

III. *Ad hoc* examples : the following example, due to Whitehead and

Spanier [5] is included only as being different in character from the others; no study of it is contemplated here. The objects of $\mathscr{S}$ are topological spaces. The maps in Hom$(X, Y; \mathscr{S})$ are equivalence classes of continuous maps, two maps being equivalent if for some n their n-fold suspensions are homotopic. Addition is defined by a process analogous to that used in defining the addition in homotopy groups. Composition is induced by the composition of continuous maps.

Additional examples will be considered below, and in a forthcoming paper.

If $\mathscr{K}$ and $\mathscr{L}$ are additive (or in fact preadditive) categories a functor $F: \mathscr{K} \to \mathscr{L}$ is said to be additive if for any $A, B \in \mathscr{K}, f, g: A \to B$,

$$F(f+g) = Ff + Fg .$$

The following important condition on an additive category is not apparently satisfied by all the reasonable examples.

(A2) (Cancellation axiom). If $i: A \to B$ and $j: B \to A$ satisfy $ji = 1: A$ then there is an object A' and maps i', j' such that (A, A', i, j, i', j') is a direct sum decomposition of B.

A category which does satisfy this condition will be called a *category with cancellation*; the examples under I above are all categories with cancellation; those under II are categories with cancellation if the class of objects contains all direct summands of its elements, which is certainly true of the class of projectives, for example. It is not clear whether the example III has cancellation or not.

A *graded preadditive category* $\mathscr{K}$ is a category in which for each $A, B \in \mathscr{K}$ the set Hom$(A, B; \mathscr{K})$ has the structure of a graded abelian group:

$$\mathrm{Hom}(A, B; \mathscr{K}) = \sum\nolimits_{k \in Z} \mathrm{Hom}_k(A, B; \mathscr{K})$$

where Z stands for the integers. The subgroup $\mathrm{Hom}_k(A, B; \mathscr{K})$ is the *homogeneous component of degree* k; its elements are the *homogeneous maps of degree* k. The structure is to satisfy an axiom (A^∞) obtained from axiom (A) by adding the homogeneity condition

$$\mathrm{Hom}_m(B, C) \otimes \mathrm{Hom}_k(A, B) \longrightarrow \mathrm{Hom}_{k+m}(A, C) .$$

In such a category, *zero-object* and *direct sum* are defined as in the ungraded case, with the additional condition that injections and projections must be homogeneous maps. If (A, A', i, j, i', j') is a direct sum decomposition then of course the degree of j is -1 times the degree of i and the degree of j' is -1 times the degree of i'.

An axiom $(A0^\infty)$, identical to (A0), and the following direct sum axiom complete the definition of a *graded additive category*.

($A1^\infty$) Any two objects of $\mathscr{K}$ have a direct sum with injections of degree 0.

The cancellation axiom in the graded case is the same as (A2) except of course that in the hypothesis the maps i, j must be taken homogeneous.

It is often important, in graded categories, to be able to shift degrees. A graded category will be said to *admit translation* if for any object A and integer k there is an object B with an equivalence $A \to B$ of degree k. It is easy to see that any graded category may be enlarged without essential change so as to admit translation.

If $\mathscr{K}$ and $\mathscr{L}$ are graded additive categories an additive functor $F : \mathscr{K} \to \mathscr{L}$ is homogeneous if for all $A, B \in \mathscr{K}$ and $k \in Z$

$$F(\mathrm{Hom}_k(A, B)) \subset \mathrm{Hom}_k(FA, FB) \text{ (or } \mathrm{Hom}_k(FB, FA)) .$$

A principal source of examples of graded categories is the following construction. If $\mathscr{K}$ is an additive category, $\mathscr{K}^\infty$ is the category having as objects the sequences $A : Z \to \mathscr{K}$ of objects of $\mathscr{K}$. If A, B are objects of $\mathscr{K}^\infty$, $\mathrm{Hom}(A, B ; \mathscr{K}^\infty)$ is defined by

$$\mathrm{Hom}_k(A, B ; \mathscr{K}^\infty) = \prod\nolimits_{m \in Z} \mathrm{Hom}(A_m, B_{m+k} ; \mathscr{K}) .$$

The *coordinates* of a map $f \in \mathrm{Hom}_k(A, B ; \mathscr{K}^\infty)$ are $f_m : A_m \to B_{m+k}$.

If $F : \mathscr{K} \to \mathscr{L}$ is a covariant additive functor a homogeneous additive functor $F^\infty : \mathscr{K}^\infty \to \mathscr{L}^\infty$ is defined by $(FA)_k = F(A_k)$ for $A \in \mathscr{K}^\infty$ and $(Ff)_k = F(f_k)$ for f a homogeneous map. If F is contravariant the definitions are $(FA)_k = F(A_{-k})$ and $(Ff)_k = F(f_{-k-r})$ for f homogeneous of degree r.

$\mathscr{K}^\infty$ is the *associated graded category* of the additive category $\mathscr{K}$. It is easy to see that $\mathscr{K}^\infty$ is a graded additive category admitting translation ; if $\mathscr{K}$ has cancellation then so also has $\mathscr{K}^\infty$. If $F : \mathscr{K} \to \mathscr{L}$ is an additive functor then F^∞, the *associated homogeneous functor* of F, is a homogeneous additive functor.

2. Monomorphisms, epimorphisms and exact sequences

In an additive category the notions of monomorphism, epimorphism and exact squence may be introduced by borrowing them from the category of abelian groups, where they are assumed to be known. Some few of the results concerning these notions go over to additive categories ; these will be considered here.

In order to construct a nontrivial theory however it is necessary either to impose restrictions on the category, making it an exact category, or to introduce additional structure. The former alternative was treated by Buchsbaum [1], the latter will be explored below, in § 3.

If $\mathscr{K}$ is an additive category a map $f: A \to B$ in $\mathscr{K}$ is a *monomorphism* if for all $C \in \mathscr{K}$ the map $\operatorname{Hom}(C, A) \to \operatorname{Hom}(C, B)$ induced by composition with f is a monomorphism in the category of abelian groups. Dually, f is an *epimorphism* if for each C the map $\operatorname{Hom}(B, C) \to \operatorname{Hom}(A, C)$ induced by composition with f is a monomorphism.

PROPOSITION 2.1. *Suppose $f: A \to B$, $g: B \to C$. Then*

(1) *If both f and g are monomorphisms (epimorphisms) then so is gf.*

(2) *If gf is a monomorphism (epimorphism) then f is a monomorphism (g is an epimorphism).*

A *short sequence* in an additive category is a diagram of the form

$$0 \longrightarrow A' \xrightarrow{a'} A \xrightarrow{a''} A'' \longrightarrow 0 .$$

It is a *short exact sequence*, abbreviated "s.e.s.", if for all C the two sequences of groups

$$\begin{array}{l} 0 \longrightarrow \operatorname{Hom}(C, A') \longrightarrow \operatorname{Hom}(C, A) \longrightarrow \operatorname{Hom}(C, A'') \\ 0 \longrightarrow \operatorname{Hom}(A'', C) \longrightarrow \operatorname{Hom}(A, C) \longrightarrow \operatorname{Hom}(A', C) \end{array}$$

with maps induced by composition with a' and a'' are exact. It follows of course that a' is a monomorphism and a'' an epimorphism.

More generally, a sequence

$$A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} A_0$$

is *exact* if there are s.e.s. $0 \longrightarrow B_j \xrightarrow{u_j} A_j \xrightarrow{v_j} B_{j-1} \longrightarrow 0$ for $j=1, \cdots, n-1$, an epimorphism $v_n: A_n \to B_{n-1}$ and a monomorphism $u_0: B_0 \to A_0$ such that $f_j = v_{j-1}u_j$.

PROPOSITION 2.2. *Suppose* $0 \longrightarrow A' \xrightarrow{a'} A \xrightarrow{a''} A'' \longrightarrow 0$ *and* $0 \longrightarrow B' \xrightarrow{b'} B \xrightarrow{b''} B'' \longrightarrow 0$ *are s.e.s. and that the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \xrightarrow{a'} & A & \xrightarrow{a''} & A'' & \longrightarrow & 0 \\ & & \downarrow{f'} & & \downarrow{f} & & & & \\ 0 & \longrightarrow & B' & \xrightarrow{b'} & B & \xrightarrow{b''} & B'' & \longrightarrow & 0 \end{array}$$

commutes. Then there is a unique map $f'': A'' \to B''$ such that the augmented diagram commutes. Dually, if f and f'' are given such that $f''a'' = b''f$ then there is a unique $f': A' \to B'$ such that $b'f' = fa'$.

This is an immediate consequence of the definition. An important special case is that in which f and f' (f and f'') are equivalences. Then the remaining map is also an equivalence.

Some special s.e.s. should be listed here. If $f: A \to B$ is an equivalence then

$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0 \longrightarrow 0$$
$$0 \longrightarrow 0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0$$

are s.e.s. If (A, A', i, j, i', j') is a direct sum decomposition of B then

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j'} A' \longrightarrow 0$$

is an s.e.s. In particular, injections of direct sums are monomorphisms, projections are epimorphisms. Equivalences are of course both.

An s.e.s. $0 \longrightarrow A' \xrightarrow{a'} A \xrightarrow{a''} A'' \longrightarrow 0$ is said to *split* if there is a map $f: A'' \to A$ such that $a''f = 1 : A''$; such a map is said to *split* the sequence. In this case $a''(1 : A - fa'') = 0$ so that $1 : A - fa'' = a'g$ for $g : A \to A'$. Then (A', A'', a', g, f, a'') is a direct sum decomposition of A. Dually, if $g : A \to A'$ and $ga' = 1 : A'$ then g *splits* the sequence, with the same consequences.

Direct sums of s.e.s. are again s.e.s.

PROPOSITION 2.3. *Suppose* $0 \longrightarrow A' \xrightarrow{a'} A \xrightarrow{a''} A'' \longrightarrow 0$ *and* $0 \longrightarrow B' \xrightarrow{b'} B \xrightarrow{b''} B'' \longrightarrow 0$ *are s.e.s. and let* $(A', B', \alpha', \overline{\alpha}', \beta', \overline{\beta}')$, $(A, B, \alpha, \overline{\alpha}, \beta, \overline{\beta})$, $(A'', B'', \alpha'', \overline{\alpha}'', \beta'', \overline{\beta}'')$ *be direct sum decompositions of* $A' + B'$, $A + B$, $A'' + B''$. *Then*

$$0 \longrightarrow A' + B' \xrightarrow{c'} A + B \xrightarrow{c''} A'' + B'' \longrightarrow 0$$

is an s.e.s., where

$$c' = \alpha\, a'\, \overline{\alpha}' + \beta\, b'\, \overline{\beta}'$$
$$c'' = \alpha'' a'' \overline{\alpha} + \beta'' b'' \overline{\beta} \,.$$

The verification is straightforward. For example, if $f : C \to A + B$ and $c''f = 0$ then $a''\overline{\alpha}f = 0$ so that $\overline{\alpha}f = a'g'$ and similarly $\overline{\beta}f = b'h$. But then $c'(\alpha'g' + \beta'h') = f$.

The first Noether isomorphism theorem has an analogue, or more properly a generalization, as well as a dual, in an arbitrary additive category.

PROPOSITION 2.4. *In the commutative diagram*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \xrightarrow{a'} & A & \xrightarrow{a''} & A'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u''} & & \\
0 & \longrightarrow & B' & \xrightarrow{b'} & B & \xrightarrow{b''} & B'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle v'} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v''} & & \\
0 & \longrightarrow & C' & \xrightarrow{c'} & C & \xrightarrow{c''} & C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

suppose the three columns are exact. If $C' = 0$ *and the first two rows are exact then* c'' *is an equivalence. Dually, if* $A'' = 0$ *and the second two rows are exact then* a' *is an equivalence.*

In the former case $vb'u' = 0$. Thus, since u' is an epimorphism, $vb' = 0$ and $v = \beta b''$ for $\beta : B'' \to C$. Since $c''\beta b'' = c''v = v''b''$ it follows that $c''\beta = v''$.

But $\beta u''a'' = \beta b''u = vu = 0$ so that $\beta = \gamma v''$ for $\gamma : C'' \to C$. Since $\gamma c''v = \gamma v''b'' = \beta b'' = v, \gamma c'' = 1 : C$. Since $c''\gamma v'' = c''\beta = v''$, $c''\gamma = 1 : C''$.

An additive functor $F : \mathscr{K} \to \mathscr{L}$ is *exact* if it takes s.e.s. into s.e.s. More generally, if $\mathfrak{S}$ is any class of s.e.s. in $\mathscr{K}$, F is *exact on* $\mathfrak{S}$ if the image of a sequence in $\mathfrak{S}$ is exact in $\mathscr{L}$.

In graded additive categories the several notions of monomorphism, epimorphism, s.e.s. are all defined analogously, with the restriction that all maps considered are homogeneous. The analogues of all the above propositions are true. They need not be stated explicitly.

If $\mathscr{K}$ is an additive category the following criteria are easily seen to hold in $\mathscr{K}^{\infty}$: a homogeneous map is a monomorphism or an epimorphism if and only if all its coordinates are ; a sequence

$$0 \longrightarrow A' \xrightarrow{a'} A \xrightarrow{a''} A'' \longrightarrow 0$$

with a' and a'' homogeneous of degrees k and 1, say, is exact if and only if each of the sequences

$$0 \longrightarrow A'_{j-k} \xrightarrow{a'_{j-k}} A_j \xrightarrow{a''_j} A''_{j+1} \longrightarrow 0$$

is an s.e.s. in $\mathscr{K}$.

3. Abelian categories

An *abelian category* is an additive category together with a subset of its maps, whose elements are to be known as *proper maps*. If a proper map is a monomorphism or an epimorphism it is a *proper monomorphism* or a *proper epimorphism*. A short exact sequence whose maps are proper is a *proper s.e.s.*; more generally, any diagram of which all maps are proper is a proper diagram.

The class of proper maps is to be subject to the following axioms:

(P0) Every identity map is proper.

(P1) If $f: B \to C$ is proper and $g: A \to B$ is a proper epimorphism then fg is proper; dually, if f is proper and $h: C \to D$ is a proper monomorphism, then hf is proper.

(P2) If fg is a proper monomorphism then g is proper; dually, if hf is a proper epimorphism then h is proper.

(P3) If $f: B \to D$ is proper there are proper s.e.s. $0 \to A \to B \to C \to 0$ and $0 \to C \to D \to E \to 0$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{a} & B & \xrightarrow{b} & C & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle 1:C} & & \\ 0 & \longleftarrow & E & \xleftarrow{d} & D & \xleftarrow{c} & C & \longleftarrow & 0 \end{array}$$

commutes.

(P4) If in the commutative diagram

$$(3.1)\qquad \begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \xrightarrow{a'} & A & \xrightarrow{a''} & A'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u''} & & \\ 0 & \longrightarrow & B' & \xrightarrow{b'} & B & \xrightarrow{b''} & B'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle v'} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v''} & & \\ 0 & \longrightarrow & C' & \xrightarrow{c'} & C & \xrightarrow{c''} & C'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

all columns and the second two rows are proper s.e.s. then the first row is also a proper s.e.s.

If the class of all maps of an additive category satisfies axioms (P3, 4), the remaining ones being of course vacuous, the category is an *exact category*. Thus the notion of abelian category generalizes that of exact category, the latter carrying, of course, the abelian structure consisting of all its maps.

This definition of exact category is easily seen to be equivalent to that of Buchsbaum [1], with the reservation that Buchsbaum does not require that an exact category have direct sums (axiom (A1) here). It should also be noted that in the exact case, i. e., that in which all maps are proper, axiom (P4) is redundant. The proof of this fact which is given in [1] seems incomplete, but others have been supplied by N. Yoneda and J. H. C. Whitehead.

Examples of abelian categories other than exact categories will be given below, especially in § 7.

In view of (P3) it is clear that an abelian structure is completely characterized by the class of proper s.e.s. A class of s.e.s. will be called *abelian* if it is the class of proper s.e.s. of an abelian structure. Axioms (P0–4) are easily translated into a characterization of abelian classes of s.e.s.

PROPOSITION 3.2. *A class* $\mathfrak{S}$ *of s.e.s. in an additive category* $\mathscr{K}$ *is abelian if and only if*

(0) *for each* $A \in \mathscr{K}$ *the s.e.s.*

$$0 \longrightarrow A \xrightarrow{1:A} A \longrightarrow 0 \longrightarrow 0, \quad 0 \longrightarrow 0 \longrightarrow A \xrightarrow{1:A} A \longrightarrow 0$$

are in $\mathfrak{S}$.

(1) *if*

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' \longrightarrow 0 \\ & & & & {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle 1:B} & & \\ 0 & \longleftarrow & B'' & \longleftarrow & B & \longleftarrow & B' & \longleftarrow & 0 \end{array}$$

commutes and both rows are in $\mathfrak{S}$ *then there is an s.e.s.* $0 \longrightarrow A' \xrightarrow{f} B \longrightarrow C'' \longrightarrow 0$ *in* $\mathfrak{S}$; *the dual also holds.*

(2) *if*

$$\begin{array}{ccccccc} 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' \longrightarrow 0 \\ & & \big\downarrow & & \big\downarrow{\scriptstyle 1:B'} & & \\ & & A & \xrightarrow{a'} & B' & & \end{array}$$

commutes and the short sequence is in $\mathfrak{S}$ *then there is a sequence* $0 \longrightarrow A' \xrightarrow{a'} A \longrightarrow A'' \longrightarrow 0$ *in* $\mathfrak{S}$; *the dual also holds.*

(4) *if the diagram* (3.1) *commutes, and all columns and the second two rows are in* $\mathfrak{S}$ *then the first row is also in* $\mathfrak{S}$.

Conditions 0, 1, 2, 4 simply express (P0, 1, 2, 4) ; axiom (P3) simply becomes a characterization of a proper map : a map is proper if it is the composition of the epimorphism of a sequence in $\mathfrak{S}$ followed by the monomorphism of a sequence in $\mathfrak{S}$.

If $\mathscr{K}$ and $\mathscr{L}$ are abelian categories an additive functor $F : \mathscr{K} \to \mathscr{L}$ is *proper* if it takes proper maps in $\mathscr{K}$ into proper maps in $\mathscr{L}$. It is *proper exact* if it takes proper s.e.s. in $\mathscr{K}$ into proper s.e.s. in $\mathscr{L}$. A proper exact functor is of course proper, but it need not be exact on s.e.s. other than proper ones.

For the remainder of this section consequences of axioms (P0–3) only will be examined.

PROPOSITION 3.3. *In an abelian category all equivalences are proper maps; also all injections and projections of direct sums are proper; also all zero-maps are proper.*

It is not asserted, and is not in general true, that compositions or sums or sums of proper maps are proper.

PROPOSITION 3.4. *The factorization of* (P3) *is unique in the following sense: If* $b' : B \to C'$ *is an epimorphism,* $c' : C' \to D$ *is a monomorphism and* $c'b' = f$ *then there is an equivalence* $\gamma : C \to C'$ *such that* $b' = \gamma b$, $c'\gamma = c$.

For $c'b'a = fa = 0$, whence $b'a = 0$ and $b' = \gamma b$. But $c'\gamma b = c'b' = f = cb$ whence $c'\gamma = c$. Dually there is a $\gamma' : C' \to C$ such that $\gamma' b' = b$, $c\gamma' = c'$. But $c(1 : C - \gamma'\gamma)b = cb - c'b' = 0$, whence $\gamma'\gamma = 1 : C$. Dually, $\gamma\gamma' = 1 : C'$.

In these circumstances it seems reasonable to refer to the factorization of (P3) as the *canonical factorization* of a proper map. The objects A, C, E, which are unique up to a transitive family of equivalences, are the *kernel*, *image* and *cokernel* of the map.

It is possible to make the kernel, image and cokernel functors on a category whose objects are the proper maps of the original category. This may be done in two ways: for each map a choice may be made among the equivalent possibilities, or the value may be taken to be the whole of the transitive equivalence system. In the former case, the functor is defined only up to natural equivalence; in the latter, it seems difficult to regard the values as lying in the original category. The problem will be avoided at this point by not considering the kernel, image and cokernel as functors. Similar situations will arise below, however, in §§ 9 and 12, where the former alternative will be chosen.

PROPOSITION 3.5. *An additive category carries abelian structures if and only if it satisfies the cancellation axiom.*

Suppose $\mathscr{K}$ is abelian, that $i : A \to B$ and $j : B \to A$, and that $ji = 1 : A$. Then j is proper and by (P3) and (3.4) there is a proper s.e.s. $0 \longrightarrow A' \xrightarrow{i'} B \xrightarrow{j} A \longrightarrow 0$. But i splits this sequence and thus leads to a direct sum decomposition of B.

If on the other hand $\mathscr{K}$ is additive and satisfies (A2) then the class of splitting s.e.s. in $\mathscr{K}$ is easily seen to be abelian.

This last structure is obviously contained in any abelian structure on $\mathscr{K}$, and will be called the *minimal* structure on $\mathscr{K}$.

The following cancellation lemma is sometimes useful.

LEMMA 3.6. *Suppose $v : G \to D$ is a proper monomorphism, $u : B \to G$ and vu is proper. Then u is proper. Dually, if u is a proper epimorphism then v is proper.*

Consider the canonical factorization of $f = vu$. Since $0 = fa = vua$ it follows that $ua = 0$ and thus that $u = wb$ for $w : C \to G$. But $vwb = cb$ so that $vw = c$ is a proper monomorphism. The conclusion follows from (3.4).

It is clear that a map is both a proper monomorphism and a proper epimorphism if and only if it is an equivalence. This suggests the terminology of *isomorphism* for the equivalences of an abelian category. Observe however that a map which is not proper may be both a monomorphism and an epimorphism, but is not an isomorphism.

The notion of a *graded abelian category* must also be defined. It consists of a graded additive category together with a set of homogeneous maps, to be called, again, proper maps, satisfying axioms (P0–4). All the statements made above about abelian categories hold also for graded abelian categories, with appropriate restrictions as to homogeneity.

If $\mathscr{K}$ is an abelian category then $\mathscr{K}^\infty$ becomes a graded abelian category under the convention that a homogeneous map of $\mathscr{K}^\infty$ is proper if and only if each of its coordinates is proper.

4. Consequences of axiom (P4)

Axioms (P0–3) for an abelian category are stated in self-dual form. It is perhaps not immediately evident, but axiom (P4) is also self-dual. More precisely, it implies its dual.

PROPOSITION 4.1. *Suppose, in the commutative diagram* (3.1), *that all columns and the first two rows are proper s.e.s. Then the third row is also a proper s.e.s.*

For $c''v = v''b''$ is a proper epimorphism, hence c'' is one also. Thus there is a proper s.e.s. $0 \longrightarrow D \xrightarrow{d} C \xrightarrow{c''} C'' \longrightarrow 0$. Since $vb'u' = 0$, $vb' = d\gamma$ for $\gamma : B' \longrightarrow D$. But by (P4), $0 \longrightarrow A' \xrightarrow{u'} B' \xrightarrow{\gamma} D \longrightarrow 0$ is then a proper s.e.s. The conclusion follows from (2.2).

If in (3.1) it is assumed that all columns and the first and third rows are proper s.e.s. it need not be the case that the second row is a proper s.e.s. (but see below, § 5). However the following result can be proved.

LEMMA 4.2. *Suppose in* (3.1) *that all columns and the first and third rows are proper s.e.s. Suppose also that* $(A, B', \alpha, \bar{\alpha}, \beta, \bar{\beta})$ *is a direct sum decomposition of* $A + B'$. *Then*

$$0 \longrightarrow A' \xrightarrow{\alpha a' - \beta u'} A + B' \xrightarrow{u\bar{\alpha} + b'\bar{\beta}} B \xrightarrow{v''b''} C'' \longrightarrow 0$$

is proper exact. Dually,

$$0 \longrightarrow A' \longrightarrow B \longrightarrow B'' + C \longrightarrow C'' \longrightarrow 0$$

is proper exact.

Let $(B, B', i, \bar{i}, i', \bar{i'})$ be a direct sum decomposition of $B + B'$. Then $(B, B', i, \bar{i} + b'\bar{i'}, ib' - i', -\bar{i'})$ is also a direct sum decomposition of $B + B'$. Moreover $\alpha a' - \beta u' : A' \to A + B'$ is a proper monomorphism, since $\bar{\alpha}(\alpha a' - \beta u') = a'$. Thus there is a proper s.e.s.

$$0 \longrightarrow A' \xrightarrow{\alpha a' - \beta u'} A + B' \xrightarrow{\delta} D \longrightarrow 0 .$$

Also

$$0 \longrightarrow A + B' \xrightarrow{iu\bar{\alpha} + i'\bar{\beta}} B + B' \xrightarrow{v\bar{i}} C \longrightarrow 0$$

is a proper s.e.s. It is exact by (2.3), and $v\bar{i}$ is certainly a proper epimorphism.

In consequence of (P4), then, the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha a' - \beta u'} & & \downarrow{\scriptstyle ib' - i'} & & \downarrow{\scriptstyle c'} & & \\
0 & \longrightarrow & A + B' & \xrightarrow{iu\bar{\alpha} + i'\bar{\beta}} & B + B' & \xrightarrow{v\bar{i}} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \bar{i} + b'\bar{i'}} & & \downarrow{\scriptstyle c''} & & \\
0 & \longrightarrow & D & \xrightarrow{\varepsilon} & B & \xrightarrow{v''b''} & C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

has its third row exact, where $\varepsilon\delta = (\bar{i} + b'\bar{i'})(iu\bar{\alpha} + i'\bar{\beta}) = u\bar{\alpha} + b'\bar{\beta}$.

PROPOSITION 4.3. *Suppose the diagram*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & X' & \xrightarrow{x'} & X & \xrightarrow{x''} & X'' & & \\
 & & {\scriptstyle \xi'}\downarrow & & {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \xi''} & & \\
 & & A' & \xrightarrow{a'} & A & \xrightarrow{a''} & A'' & \longrightarrow & 0 \\
 & & {\scriptstyle t'}\downarrow & & {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t''} & & \\
0 & \longrightarrow & B' & \xrightarrow{b'} & B & \xrightarrow{b''} & B'' & & \\
 & & {\scriptstyle \eta'}\downarrow & & {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \eta''} & & \\
 & & Y' & \xrightarrow{y'} & Y & \xrightarrow{y''} & Y'' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\tag{4.4}
$$

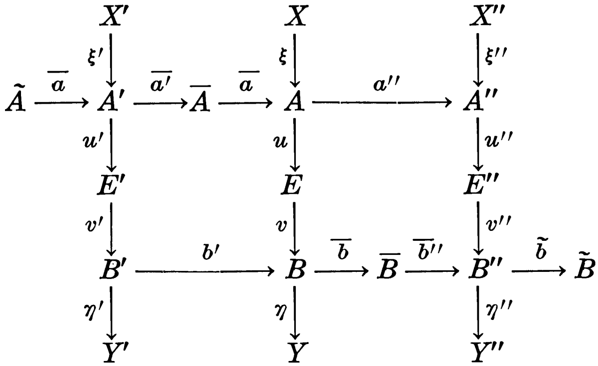
commutes and that all columns and the two center rows are proper exact. Then the top and bottom rows are proper exact, and in addition there is a natural equivalence of the cokernel of x'' with the kernel of y', so that $X' \to X \to X'' \to Y' \to Y \to Y''$ is proper exact.

The cokernel of x'' and the kernel of y' are naturally equivalent as functors on a category whose objects are diagrams of the form (4.4). The naturality will however become clear in the proof without formal consideration of this category.

In a structure satisfying axioms (P0–3) this condition is equivalent to (P4). That it implies (P4) is easily seen, since (3.1), with all columns and the second two rows proper exact, becomes a special case of (4.4) if an extra row of zeros is added at the bottom. But then, in (3.1), $A' \to A \to A'' \to 0$ is proper exact. Since a' is clearly a proper monomorphism, the first row of (3.1) is a proper s.e.s.

The proof of (4.3) is as follows.

First recall that a', t', t, t'', b'' are proper maps and introduce their canonical factorizations, producing from (4.4) the following diagram,



,

in which all maps appear in proper s.e.s.

Now $\eta b'v'u' = \eta ta' = 0$, so that $\eta b'v' = 0$ and $b'v' = v\varepsilon'$ for $\varepsilon' : E \to E$, which must of course be a proper monomorphism. Moreover $v\varepsilon'u' = b'v'u' = vua'$ so that $\varepsilon'u' = u\bar{a}\bar{a}'$.

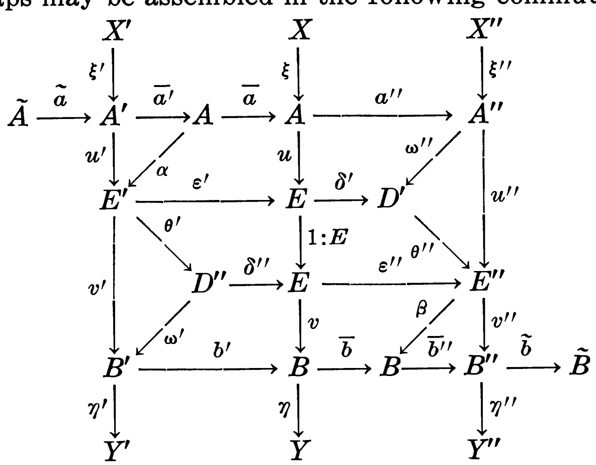
Dually there is a proper epimorphism $\varepsilon'' : E \to E''$ such that it may be introduced into the above diagram without disturbing commutativity.

Since $b'v'u'\tilde{a} = ta'\tilde{a} = 0$ it follows that $u'\tilde{a} = 0$ and thus that $u' = \alpha\bar{a}'$ for $\alpha : \bar{A} \to E'$, which must be a proper epimorphism. Dually, there is a $\beta : E'' \to \bar{B}$, which must be a proper monomorphism, such that $\bar{b}''\beta = v''$.

Now ε' and ε'' appear in proper s.e.s. $0 \longrightarrow E' \xrightarrow{\varepsilon'} E \xrightarrow{\delta'} D' \longrightarrow 0$ and $0 \longrightarrow D'' \xrightarrow{\delta''} E \xrightarrow{\varepsilon''} E'' \longrightarrow 0$. But $\varepsilon''\varepsilon'u' = u''a''a' = 0$. Thus there is a proper monomorphism $\theta' : E' \to D''$ such that $\delta''\theta' = \varepsilon'$ and a proper epimorphism $\theta'' : D' \to E''$ such that $\theta''\delta' = \varepsilon''$.

Finally, $\delta' u\bar{a}\bar{a}' = \delta'\varepsilon'u' = 0$ so that $\delta'u\bar{a} = 0$ and $\delta'u = \omega''a''$ for a proper epimorphism $\omega'' : A'' \to D'$ and dually $v\delta'' = b'\omega'$ for a proper monomorphism $\omega' : D'' \to B'$. Clearly $\theta''\omega'' = u''$ and $\omega'\theta' = v'$.

These maps may be assembled in the following commutative diagram:



From this, using (P4), the following diagrams, all of which commute and have proper exact rows and columns, may in turn be extracted.

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \tilde{A} & \longrightarrow & X' & \longrightarrow & \overline{X'} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \tilde{A} & \longrightarrow & A' & \longrightarrow & \overline{A} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & E' & \longrightarrow & E' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \qquad \begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \bar{\bar{X}}' & \longrightarrow & X & \longrightarrow & \bar{\bar{X}}'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \overline{A} & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & D' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E'' & \longrightarrow & E'' & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \bar{\bar{B}} & \longrightarrow & B'' & \longrightarrow & \tilde{B} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \bar{\bar{Y}}'' & \longrightarrow & Y'' & \longrightarrow & \tilde{B} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \qquad \begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & D'' & \longrightarrow & E & \longrightarrow & E'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & \bar{\bar{B}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \bar{\bar{Y}}' & \longrightarrow & Y & \longrightarrow & \bar{\bar{Y}}'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \bar{\bar{X}}'' & \longrightarrow & X'' & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \bar{\bar{X}}'' & \longrightarrow & A'' & \longrightarrow & D' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & E'' & \longrightarrow & E'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \qquad \begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E' & \longrightarrow & E' & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & D'' & \longrightarrow & B' & \longrightarrow & \bar{\bar{Y}}' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G & \longrightarrow & Y' & \longrightarrow & \bar{\bar{Y}}' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

But

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E' & \longrightarrow & E' & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D'' & \longrightarrow & E & \longrightarrow & E'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F & \longrightarrow & D' & \longrightarrow & E'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

also has proper exact rows and columns so that, up to natural equivalence, $F = G$. Since x', x'', y', y'' are determined uniquely by commutativity they must be the compositions $X' \to \overline{X}' \to X$, $X \to \overline{X}'' \to X''$, $Y' \to \overline{Y}' \to Y$, $Y \to \overline{Y}'' \to Y''$ respectively, which completes the proof.

A more explicit description of the map $\Delta : X'' \to Y'$ of the above proposition is given by the following lemma.

LEMMA 4.4. *Suppose, in the situation of* (4.3), *that* $f: W \to A$, $f'': W \to A''$ *and* $a''f = \xi''f''$. *Then there is a unique map* $\varphi : W \to B'$ *such that* $b'\varphi = tf$, *and, for this map,* $\eta'\varphi = \Delta f''$.

The key to this result is $uf : W \to E$. On the one hand $\delta' uf = \omega''\xi''f''$. On the other, $\varepsilon''uf = 0$ so that $uf = \delta''\psi$ for $\psi : W \to D''$. But then $\varphi = \omega'\psi : W \to B'$ and referring to the last three diagrams above, $\Delta f'' = \eta'\varphi$.

5. Projectives, injectives, closure

If $\mathscr{K}$ is an additive category and $\mathfrak{S}$ is a class of short sequences in $\mathscr{K}$ an object $X \in \mathscr{K}$ is *projective with respect to* $\mathfrak{S}$ or $\mathfrak{S}$*-projective,* if for every $0 \to A' \to A \to A'' \to 0$ in $\mathfrak{S}$ the induced sequence of groups

$$(5.1) \quad 0 \longrightarrow \mathrm{Hom}(X, A') \longrightarrow \mathrm{Hom}(X, A) \longrightarrow \mathrm{Hom}(X, A'') \longrightarrow 0$$

is exact.

It is clear that a direct sum of two $\mathfrak{S}$-projectives is $\mathfrak{S}$-projective. If X is $\mathfrak{S}$-projective, $i : Y \to X$, $j : X \to Y$ and $ji = 1 : Y$ then Y is also $\mathfrak{S}$-projective.

The *projective closure* $\mathrm{Cl}\mathfrak{S}$ of a class $\mathfrak{S}$ of short sequences is the class of all short sequences $0 \to A' \to A \to A'' \to 0$ such that for every $\mathfrak{S}$-projective X the sequence (5.1) is exact. A class of short sequences is closed if it is equal to its projective closure.

The notions of $\mathfrak{S}$-*injective*, the *injective closure* $\mathrm{Cl}^*\mathfrak{S}$, and *injectively closed* set of short sequences, are defined dually.

PROPOSITION 5.2. *If* $\mathfrak{S}$ *and* $\mathfrak{S}'$ *are classes of short sequences in an additive category then*

(1) $\mathrm{Cl}\mathfrak{S} \supset \mathfrak{S}$

(2) $\mathrm{Cl}(\mathfrak{S} \cup \mathfrak{S}') = \mathrm{Cl}\mathfrak{S} \cup \mathrm{Cl}\mathfrak{S}'$

(3) $\mathrm{Cl}(\mathrm{Cl}\mathfrak{S}) = \mathrm{Cl}\mathfrak{S}$.

Analogous properties hold for the injective closure.

The verification is straightforward. The assertion is that Cl is a closure operation in the set of short sequences, except for the fact that the empty set is not projectively closed.

A class $\mathfrak{S}$ of short sequences is, simply, *closed*, if $\mathfrak{S} = \mathrm{Cl}\mathfrak{S} \cap \mathrm{Cl}^*\mathfrak{S}$. In particular, if $\mathfrak{S}$ is either projectively or injectively closed, then it is closed.

PROPOSITION 5.3. *If* $\mathfrak{S}$ *is a closed class of short sequences, and in the commutative diagram* (3.1) *all columns and the first two rows are in* $\mathfrak{S}$, *then the third row is also in* $\mathfrak{S}$. *If all columns and the first and third rows are in* $\mathfrak{S}$ *and* $b''b' = 0$ *then the second row is also in* $\mathfrak{S}$.

This is just an application of (P4) in the category of abelian groups and homomorphism, as regards the first statement. The second follows from the corresponding property of the category of abelian groups.

PROPOSITION 5.4. *If* $\mathscr{K}$ *is an additive category and* $\mathfrak{S}$ *a class of s.e.s. in* $\mathscr{K}$ *which satisfies conditions* (0, 1, 2) *of Proposition* (3.2), *then if* $\mathfrak{S}$ *is closed it also satisfies condition* (4), *and is thus abelian.*

This is just a corollary of (5.3).

A *projective* of an abelian category is an object which is projective with respect to the class of proper s.e.s. of the category. Clearly X is projective if and only if $\mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B)$ is an epimorphism for every proper epimorphism $A \to B$. *Injectives* of abelian categories are defined dually.

An abelian category is *closed*, *projectively closed*, or *injectively closed* if the class of s.e.s. possesses one of these properties. Proposition 5.4 asserts that axioms (P0–3), together with closure, characterize a closed abelian category. As a practical matter, this may be important in verifying the fact that a structure is abelian.

The second assertion of (5.3) leads to the following corollary.

PROPOSITION 5.5. *If in a closed abelian category the diagram* (3.1) *commutes, and all columns and the first and third rows are proper s.e.s., and in addition* $b''b' = 0$, *then the second row is a proper s.e.s.*

An abelian category has *enough projectives* if for every object A there is a proper epimorphism $X \to A$ with X projective. The dual notion is that of having *enough injectives.*

LEMMA 5.6. *In an abelian category with enough projectives a sequence*

$$A_n \xrightarrow{f_n} A_{n-1} \longrightarrow \cdots \xrightarrow{f_1} A_0 \longrightarrow 0$$

is proper exact if and only if for any projective X the induced sequence

$$\mathrm{Hom}(X, A_n) \longrightarrow \cdots \longrightarrow \mathrm{Hom}(X, A_0) \longrightarrow 0$$

is exact. In a category with enough injectives, the dual result holds.

The necessity is of course clear. The sufficiency is proved by induction on n. For $n = 1$ the assertion is just that f_1 is a proper epimorphism. But suppose $\varphi : X \to A_0$ is a proper epimorphism with X projective. Then $\varphi = f_1\Phi$ for some $\Phi : X \to A_1$ and the assertion follows.

Suppose the result is true for shorter sequences. Then if the sequence of $\mathrm{Hom}(X, A_i)$ is exact for all projective X it follows that $A_{n-1} \to \cdots \to A_0 \to 0$ is proper exact and f_{n-1} has the canonical factorization appearing in the following commutative diagram :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{u} & A_{n-1} & \xrightarrow{v} & C & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle 1:A_{n-1}} & & \downarrow{\scriptstyle w} & & \\ & & & & A_{n-1} & \xrightarrow{f_{n-1}} & A_{n-2} & & . \end{array}$$

Here the top row is a proper s.e.s. and w is a proper monomorphism.

Now for a proper epimorphism $\varphi : X \to A_n$ with X projective $f_{n-1}f_n\varphi = 0$ so that $f_{n-1}f_n = 0$. Thus $vf_n = 0$ and $f_n = u\bar{f}$ for $\bar{f} : A_n \to B$. It is enough to show that $\bar{f}$ is a proper epimorphism.

But suppose $\varphi : X \to B$ is a proper epimorphism with X projective. Then $f_{n-1}u\varphi = 0$ so that $u\varphi = u\bar{f}\Phi$ for some $\Phi : X \to A_n$. But then $\varphi = \bar{f}\Phi$, which completes the proof.

PROPOSITION 5.7. *If an abelian category has enough projectives* (*injectives*) *it is projectively* (*injectively*) *closed.*

This is just the case $n = 3$, $A_n = 0$ of the preceding lemma. It should be observed that this proposition implies that in the presence of enough projectives axiom (P4), for an abelian category, is redundant.

6. The category $\mathscr{K}^S$

If $\mathscr{K}$ is an abelian category the *category* $\mathscr{K}^S$ *of proper s.e.s. in* $\mathscr{K}$

has as objects the proper s.e.s. of $\mathscr{K}$; if $\mathbf{A} = (0 \to A' \to A \to A'' \to 0)$ and $\mathbf{B} = (0 \to B' \to B \to B'' \to 0)$ are objects of $\mathscr{K}^S$ a map $\mathbf{f}: \mathbf{A} \to \mathbf{B}$ is a sequence $(\mathbf{A}, \mathbf{B}; f', f, f'')$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \xrightarrow{a'} & A & \xrightarrow{a''} & A'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & B' & \xrightarrow{b'} & B & \xrightarrow{b''} & B'' & \longrightarrow & 0 \end{array} \tag{6.1}$$

is a commutative diagram in $\mathscr{K}$. Where no confusion is possible $(\mathbf{A}, \mathbf{B}; f', f, f'')$ may be abbreviated as (f', f, f''). Composition and addition of maps in $\mathscr{K}^S$ being defined in terms of the corresponding operations in $\mathscr{K}$, $\mathscr{K}^S$ has at least the structure of a preadditive category.

If also $\mathbf{C} = (0 \to C' \to C \to C'' \to 0) \in \mathscr{K}^S$ a sequence

$$0 \longrightarrow \mathbf{A} \xrightarrow{\mathbf{u}} \mathbf{B} \xrightarrow{\mathbf{v}} \mathbf{C} \longrightarrow 0$$

in $\mathscr{K}^S$ is a proper s.e.s. in $\mathscr{K}^S$ if the diagram 3.1 in $\mathscr{K}$ has proper exact columns.

PROPOSITION 6.2. *$\mathscr{K}^S$ is an abelian category.*

It is first necessary to verify the fact that it is an additive category. But $0 \to 0 \to 0 \to 0 \to 0$ is clearly a zero-object. To see that the direct sum axiom is satisfied suppose that $\mathbf{A}$ and $\mathbf{B}$, as above, are in $\mathscr{K}^S$. Then by (2.3), $0 \to A' + B' \to A + B \to A'' + B'' \to 0$ is an s.e.s. in $\mathscr{K}$. On the other hand, $0 \to A' + B' \to A + B' \to A'' \to 0$ is a proper s.e.s. in $\mathscr{K}$, so that $A' + B' \to A + B'$ is a proper monomorphism. Similarly $A + B' \to A + B$ is a proper monomorphism. Thus $A' + B' \to A + B$ is a proper monomorphism and the s.e.s. $0 \to A' + B' \to A + B \to A'' + B'' \to 0$ is proper. But it is clearly a direct sum of $\mathbf{A}$ and $\mathbf{B}$.

It remains to see that the class designated as proper s.e.s. in $\mathscr{K}^S$ is abelian. Conditions (0, 1, 2) of Proposition 3.2 are verified in straightforward fashion; for (1) and (2) the concluding step in each case is provided by (P4) in $\mathscr{K}$. Condition (4) in $\mathscr{K}^S$ asserts exactness in a "cubical" diagram in $\mathscr{K}$; it is easily verified by applying (P4), in $\mathscr{K}$, to the several plane sections of the diagram.

Covariant functors $S', S, S'' : \mathscr{K}^S \to \mathscr{K}$ are defined by

$$\begin{array}{lll} S'\mathbf{A} = A' & S\mathbf{A} = A & S''\mathbf{A} = A'' \\ S'\,\mathbf{f} = f' & S\,\mathbf{f} = f & S''\,\mathbf{f} = f'' \end{array}$$

where $\mathbf{A}$, $\mathbf{f}$ are described above. These are of course proper exact functors. In addition, natural transformations $s' : S' \to S$ and $s'' : S \to S''$ are defined by

$$s'\mathbf{A} = a' \qquad s''\mathbf{A} = a'' .$$

LEMMA 6.3. *If* $\mathbf{X} = (0 \longrightarrow X' \xrightarrow{x'} X \xrightarrow{x''} X'' \longrightarrow 0) \in \mathscr{K}^S$ *and* X', X'' (*and hence* X) *are projective* (*injective*) *in* $\mathscr{K}$ *then* $\mathbf{X}$ *is projective* (*injective*) *in* $\mathscr{K}^S$.

For, referring to (6.1), suppose $\mathbf{f}$ is a proper epimorphism and let $\mathbf{g} = (g', g, g'') : \mathbf{X} \to \mathbf{B}$. Then there is a $\theta : X'' \to A$ such that $f''a''\theta = g''$. Now $b''(g - f\theta x'') = g''x'' - f''a''\theta x'' = 0$ so that $g - f\theta x'' = b'f\varphi$ for $\varphi : X \to A'$.

But $\mathbf{h} = (\varphi x', \theta x'' + a'\varphi, a''\theta) : \mathbf{X} \to \mathbf{A}$ and $\mathbf{fh} = \mathbf{g}$.

PROPOSITION 6.4. *If* $\mathscr{K}$ *is projectively* (*injectively*) *closed then so is* $\mathscr{K}^S$; *also if* $\mathscr{K}$ *is closed then so is* $\mathscr{K}^S$.

Suppose first that $\mathscr{K}$ is projectively closed. If $0 \to \mathbf{A} \to \mathbf{B} \to \mathbf{C} \to 0$ is a sequence in $\mathscr{K}^S$ such that

$$(6.5) \qquad 0 \longrightarrow \mathrm{Hom}(\mathbf{X}, \mathbf{A}) \longrightarrow \mathrm{Hom}(\mathbf{X}, \mathbf{B}) \longrightarrow \mathrm{Hom}(\mathbf{X}, \mathbf{C}) \longrightarrow 0$$

is exact for every projective $\mathbf{X} \in \mathscr{K}^S$, then in particular it is exact for $\mathbf{X}$ of the form $0 \longrightarrow X \longrightarrow X \longrightarrow 0 \longrightarrow 0$ with $X \in \mathscr{K}$ projective. But for such $\mathbf{X}$, $S' : \mathrm{Hom}(\mathbf{X}, \mathbf{A}) \approx \mathrm{Hom}(X, A')$. Thus $0 \to A' \to B' \to C' \to 0$ is a proper s.e.s. in $\mathscr{K}$. Also (6.5) is exact for $\mathbf{X}$ of the form $0 \to 0 \to X \to X \to 0$. But for such $\mathbf{X}$, $S : \mathrm{Hom}(\mathbf{X}, \mathbf{A}) \approx \mathrm{Hom}(X, A)$. Thus $0 \to A \to B \to C \to 0$ is a proper s.e.s. in $\mathscr{K}$. The conclusion follows from (P4) in $\mathscr{K}$.

If $\mathscr{K}$ is injectively closed the result is of course dual. The argument is not essentially different when $\mathscr{K}$ is simply closed ; it may be omitted.

PROPOSITION 6.5. *If* $\mathscr{K}$ *has enough projectives* (*injectives*) *then so has* $\mathscr{K}^S$. *In this case every projective* (*injective*) *is of the form described in* Lemma 6.3.

For if $\mathbf{A} \in \mathscr{K}^S$ there are proper epimorphisms $\xi' : X' \to A'$ and $\theta : X'' \to A$ with X', X'' projective. Suppose $(X', X'', x', \bar{x}', x', \bar{x}'')$ is a direct sum decomposition of X. Then $\mathbf{X} = (0 \longrightarrow X' \xrightarrow{x} X \xrightarrow{\bar{x}} X'' \longrightarrow 0)$ is projective in $\mathscr{K}^S$. But $(\xi', \theta\bar{x}'', a''\theta) : \mathbf{X} \to \mathbf{A}$ is a proper epimorphism.

If $\mathbf{A}$ is projective it is a direct summand of $\mathbf{X}$, and is thus of the same form.

If $\mathscr{K}$ is an abelian category a *connected covariant* (*contravariant*) *functor* (T, τ) on $\mathscr{K}$ is a covariant (contravariant) functor $T : \mathscr{K} \to \mathscr{L}$ together with a natural transformation $\tau : TS'' \to TS'$ ($\tau : TS' \to TS''$) of the composed functors on $\mathscr{K}^S$. Thus if $\mathbf{A} = (0 \to A \to A' \to A'' \to 0)$ is a proper s.e.s. in $\mathscr{K}$ and (T, τ) is a covariant connected functor on $\mathscr{K}$ the triangular diagram

$$\begin{array}{ccc} TA' & \longrightarrow & TA \\ & \nwarrow & \downarrow \\ & & TA'' \end{array}$$

is defined. If T is contravariant the arrows are of course reversed.

The most important case is that in which $\mathscr{K}$ is an additive category and T an additive functor. The following anticommutation condition then holds.

PROPOSITION 6.6. *If* (T, τ) *is a covariant connected additive functor on* $\mathscr{K}$, *and if Diagram* (3.1) *in* $\mathscr{K}$ *has proper exact rows and columns then*

$$\begin{array}{ccc} TC'' & \longrightarrow & TC' \\ \downarrow & & \downarrow \\ TA'' & \longrightarrow & TA' \end{array}$$

anticommutes, the maps being the appropriate connecting homomorphisms, e. g., $\tau(0 \to A'' \to B'' \to C'' \to 0) : TC'' \to TA''$. *If in addition* $A' = 0$ *then*

$$\begin{array}{ccc} TC'' & \longrightarrow & TA'' \\ \downarrow & & \downarrow{\scriptstyle T(ua''^{-1})} \\ TC' & \underset{T(b'v'^{-1})}{\longrightarrow} & TB \end{array}$$

anticommutes. If instead $C'' = 0$ *then*

$$\begin{array}{ccc} TB & \overset{T(u''^{-1}b'')}{\longrightarrow} & TA'' \\ {\scriptstyle T(c'^{-1}v)}\downarrow & & \downarrow \\ TC & \longrightarrow & TA' \end{array}$$

anticommutes.

The proof, using Lemma 4.2, is identical with that of [2, III] and will be omitted.

7. Relations between abelian structures; examples

If $\mathscr{K}$ is an additive category with cancellation it has at least one and perhaps several abelian structures. These are partially ordered by the inclusion relation on the classes of proper maps or, equivalently, on the classes of proper s.e.s. There is always a minimum structure; there may also be a maximum one; for example this is the case if the class of all s.e.s. is abelian or, *a fortiori*, if the category is exact.

PROPOSITION 7.1. *The collection of abelian structures on an additive category is inductive. Also, the intersection of any collection of abelian* *structures is abelian.*

This is a direct consequence of the axioms. In particular, if there is a maximum structure the collection of abelian structures is a lattice.

PROPOSITION 7.2. *If $\mathfrak{S}$ is an abelian class of s.e.s. in $\mathscr{K}$ and $\mathfrak{T}$ is a closed class of short sequences in $\mathscr{K}$ then $\mathfrak{S} \cap \mathfrak{T}$ is abelian.*

This is just a matter of applying Proposition 5.3. For example, referring to condition (1) of Proposition 3.2, suppose the two s.e.s. are both in $\mathfrak{S} \cap \mathfrak{T}$. Then the diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & A' & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C' & \longrightarrow & D & \longrightarrow & B'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

has all its rows and columns in $\mathfrak{S}$. Thus $0 \to C' \to D \to B'' \to 0$ is in $\mathfrak{T}$. But then $0 \to A' \to B \to D \to 0$ is also in $\mathfrak{T}$.

An additive functor $F: \mathscr{K} \to \mathscr{L}$ also maps short sequences in $\mathscr{K}$ into short sequences in $\mathscr{L}$. If $\mathfrak{T}$ is a set of short sequences in $\mathscr{L}$ the expression $F^{-1}(\mathfrak{T})$ is to be understood in the light of this observation.

PROPOSITION 7.3. *If $\mathfrak{S}$, $\mathfrak{T}$ are abelian classes of s.e.s. in $\mathscr{K}$ and $\mathscr{L}$ respectively and $F: \mathscr{K} \to \mathscr{L}$ is exact on $\mathfrak{S}$ then $\mathfrak{S} \cap F^{-1}(\mathfrak{T})$ is an abelian class of s.e.s. on $\mathscr{K}$.*

The proof is straightforward. $\mathfrak{S} \cap F^{-1}(\mathfrak{T})$ gives the abelian structure *induced* by F. Evidently if F is proper exact it induces the original structure in $\mathscr{K}$.

These last two propositions may evidently be used to produce new abelian structures on abelian categories. Proposition 7.2 asserts that a relatively closed substructure of an abelian structure is abelian. In particular, then, if $\mathscr{K}$ is an abelian category and $\mathfrak{A}$ is a set of objects in $\mathscr{K}$ there is a maximal substructure such that all the objects of $\mathfrak{A}$ are projective in the substructure. If they are not all projective in $\mathscr{K}$ the new structure must of course be distinct from the original one. For example in the category of finitely generated abelian groups and homomorphisms a proper substructure is generated by demanding that all the p-groups, say, be projective. A map in the substructure is then proper if the p-primary summand of the image is a direct summand of the range and a direct factor of the domain ; the substructure is thus different also from the minimum structure.

Proposition 7.3 is illustrated in the following example which is due to S. Eilenberg, and is implicit in [2, VIII, § 3]. Suppose Λ and Γ are rings with units and $\varphi : \Gamma \rightarrow \Lambda$ is a unitary ring homomorphism. If $\mathscr{M}(\Lambda)$ and $\mathscr{M}(\Gamma)$ are the categories of unitary left modules φ may be used to define an exact covariant functor $\Phi : \mathscr{M}(\Lambda) \rightarrow \mathscr{M}(\Gamma)$, a Λ-module being regarded as a Γ-module with operations defined by φ. Using the minimum structure in $\mathscr{M}(\Gamma)$, this functor induces in $\mathscr{M}(\Lambda)$ a substructure of the exact structure known as the *φ-relative structure*. This is the structure used to define the relative derived functors, and in particular the relative Tor and Ext functors, in the manner to be described below.

This relative structure also exemplifies Proposition 7.2, since it has enough projectives. For if A is a Λ-module then $\Lambda \otimes_{\Gamma} A$ is a relative projective which has a proper epimorphism onto A, given by $\lambda \otimes a \rightarrow \lambda a$.

Both of the above examples arise as substructures of exact categories. But there are of course examples for which this is not the case. The underlying additive category of $\mathscr{K}^S$ is not exact even if $\mathscr{K}$ is. This may be seen as follows. Let $\mathbf{A} = (0 \longrightarrow A' \xrightarrow{a'} A \xrightarrow{a''} A'' \longrightarrow 0)$ be an s.e.s. in $\mathscr{K}$ which does not split, and let $\mathbf{B}$ be the s.e.s. $(0 \longrightarrow 0 \longrightarrow A \xrightarrow{1:A} A \longrightarrow 0)$. Then $(0, 1 : A, a'') : \mathbf{B} \rightarrow \mathbf{A}$ is both a monomorphism and an epimorphism, but is obviously not an equivalence.

A final example may be adduced as differing in spirit from the preceding ones. In the category of (real or complex) Banach spaces and continuous linear maps define a map to be proper if its image is closed. This is easily seen to give rise to an abelian structure. In fact this structure has enough projectives, namely the spaces $l_1(J)$ where J is a discrete set, i. e., the space of maps $w : J \rightarrow R$ (say) such that

$$|w| = \sum_{j \in J} |w_j| < \infty \ .$$

The abelian structure however is not the minimum one ; F. J. Murray has shown [4] that in the spaces L_p, $p \neq 2$, there are closed subspaces without closed complements, which is to say that the corresponding s.e.s. do not split.

Once more, this is not a substructure of an exact structure. For instance, if H is a separable Hilbert space with basis $x_1, x_2, \cdots$ then $f : H \rightarrow H$, with $fx_n = x_n/n$, is both a monomorphism and an epimorphism, but is not an equivalence.

It is to be observed here that the composition of two proper maps is not in general proper. For instance if V, W are closed subspaces of X such that $V + W$ is dense in X then the inclusion $V \to X$ and the projection $X \to X/W$ are proper maps whose composition is not proper.

Similar phenomena occur, in fact, in all the above examples.

8. Objects-with-derivation

If $\mathscr{K}$ is an additive category the category $d\mathscr{K}$ of *objects-with-derivation* in $\mathscr{K}$ has as its objects the pairs (A, d) where A is an object of $\mathscr{K}$, $d : A \to A$ in $\mathscr{K}$ and $d^2 = 0$. The map d is the *derivation* of (A, d). The group $\operatorname{Hom}((A, d), (A', d'); d\mathscr{K})$ consists of the triples $((A, d), (A', d'), f)$ where $f : A \to A'$ in $\mathscr{K}$ and $d'f = fd$. Composition and addition of maps in $d\mathscr{K}$ are defined by the corresponding operations in $\mathscr{K}$. This gives $d\mathscr{K}$ the structure of an additive category; it is easy to see that if $\mathscr{K}$ has cancellation then so has $d\mathscr{K}$.

Now $(A, d) \to A$, $((A, d), (A', d'), f) \to f$ defines an additive functor from $d\mathscr{K}$ to $\mathscr{K}$. In conformity with the usual ambiguous convention there will be no notation for this functor: if A is an object of $d\mathscr{K}$ it will also stand for the underlying object of $\mathscr{K}$; the map $((A, d), (A', d'), f)$ will usually be denoted by f. Thus it will sometimes be necessary to distinguish between "$f : A \to A'$ in $\mathscr{K}$" and the stronger statement "$f : A \to A'$ in $d\mathscr{K}$." In addition, $(A, d) \to d$ is a natural transformation of this functor into itself for which again an ambiguous notation will be used. The derivation of an object $A \in d\mathscr{K}$ will be denoted by d_A or sometimes simply by d, d' and so forth.

If $A \in \mathscr{K}$ and (A, A, i, j, i', j') is a direct sum decomposition of $A + A$ then $(A + A, i'j)$ is an object of $d\mathscr{K}$, which will be denoted by A^x. Such objects are called *null-objects*.

LEMMA 8.1. *If $A \in \mathscr{K}$, $B \in d\mathscr{K}$ then $f \to fi$ defines an isomorphism of $\operatorname{Hom}(A^x, B; d\mathscr{K})$ with $\operatorname{Hom}(A, B; \mathscr{K})$. Dually $f \to j'f$ defines an isomorphism of $\operatorname{Hom}(B, A^x; d\mathscr{K})$ with $\operatorname{Hom}(B, A; \mathscr{K})$.*

The inverse, in the first case, is $g \to gj + dgj'$.

If A and A' are objects of $d\mathscr{K}$ and $f : A \to A'$ in $\mathscr{K}$ then $d'f + fd : A \to A'$ in $d\mathscr{K}$. Such maps are said to be *nullhomotopic*; they form a subgroup of $\operatorname{Hom}(A, A'; d\mathscr{K})$. The factor group, the group of *homotopy classes* of A into A' will be denoted by $\mathfrak{Hom}(A, A')$. Two maps are *homotopic* if they belong to the same class in $\mathfrak{Hom}(A, A')$. The notion of *homotopy equivalence* of objects of $d\mathscr{K}$ is defined in the usual way.

PROPOSITION 8.2. *A map of $d\mathscr{K}$ is nullhomotopic if and only if it can* *be factored through a null-object.*

For suppose $g : \to A^x$ and $h: A^x \to C'$. Then $d'(hij'g) + (hij'g)d = hi'jij'g + hij'i'jg = hg$. On the other hand if $f : C \to A$ in $\mathscr{K}$ then $j + dj' : A^x \to A$ and $ifd' + i'f : C \to A^x$ are both in $d\mathscr{K}$ and have composition $df + fd$.

If $\mathscr{K}$ is an abelian category then $d\mathscr{K}$ can also be given an abelian structure.

PROPOSITION 8.3. *If $\mathscr{K}$ is an abelian category and a map (A, A', f) of $d\mathscr{K}$ is defined to be proper whenever f is proper in $\mathscr{K}$ then $d\mathscr{K}$ has structure of an abelian category.*

Axioms (P0–2) and (P4) are trivially verified. For (P3) suppose $f : B \to D$ in $d\mathscr{K}$ with f proper in $\mathscr{K}$ and consider the canonical factorization of f in $\mathscr{K}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{a} & B & \xrightarrow{b} & C & \longrightarrow & 0 \\ & & & & {\scriptstyle f}\downarrow & & \downarrow & & \\ 0 & \longleftarrow & E & \xleftarrow{e} & D & \xleftarrow{c} & C & \longleftarrow & 0 \end{array} .$$

Now $cbd_Ba = fd_Ba = d_Dfa = 0$ so that $bd_Ba = 0$ and $bd_B = d_Cb$ for $d_C : C \to C$. But $d_C^2b = bd_B^2 = 0$ so that $d_C^2 = 0$ and $(C, d_C) \in d\mathscr{K}$ and $b : B \to C$ in $d\mathscr{K}$. On the other hand, $cd_Cb = cbd_B = d_Df = d_Dcb$ so that $cd_C = d_Dc$ and $c : C \to D$ in $d\mathscr{K}$. The rest of the argument is similar.

PROPOSITION 8.4. *If X is a projective (injective) of $\mathscr{K}$ then X^x is a projective (injective) of $d\mathscr{K}$; if $\mathscr{K}$ is closed (projectively, injectively closed) then so is $d\mathscr{K}$; if $\mathscr{K}$ has enough projectives (injectives) then so has $d\mathscr{K}$.*

These are just applications of Lemma 8.1.

In the graded case it is usual to demand that the derivations be homogeneous of degree 1 or -1 ; the latter alternative will be chosen here. If $\mathscr{K}$ is a graded additive category, $\partial\mathscr{K}$ will denote the category having as objects the pairs (A, ∂) where A is an object of $\mathscr{K}$ and $\partial : A \to A$ in $\mathscr{K}$ is a homogeneous map of degree -1 such that $\partial^2 = 0$. Hom $((A, \partial), (A', \partial') ; \partial\mathscr{K})$ is a graded group : $\text{Hom}_k((A, \partial), (A', \partial') ; \partial\mathscr{K})$ has as elements the triples $((A, \partial), (A', \partial'), f)$ where $f : A \to A'$ is a homogeneous map of degree k in $\mathscr{K}$ such that $\partial' f = (-1)^k f\partial$. Composition and addition being defined by the corresponding operations in $\mathscr{K}$, $\partial\mathscr{K}$ has the structure of an additive category.

The nullhomotopic maps of A into A' in $\partial\mathscr{K}$ are a sub-graded-group with homogeneous component of degree k composed of maps $\partial' f + (-1)^k f\partial$, for $f : A \to A'$ in $\mathscr{K}$ of degree $k+1$. The factor group, which is the group of homotopy classes of maps of A into A', is a graded group also,

$\mathfrak{Hom}(A, A') = \sum_k \mathfrak{Hom}_k(A, A')$.

The observations made above with regard to the category $d\mathscr{K}$ all have obvious cognates in $\partial\mathscr{K}$; proper attention being paid to degrees.

9. Proper derivations ; homology

Even if $\mathscr{K}$ is an abelian category an object in $d\mathscr{K}$ cannot be expected to have homology : an obvious requirement is that its derivation be a proper map. Such objects, and the maps of $d\mathscr{K}$ connecting them, constitute the subcategory $d^P\mathscr{K}$ of $d\mathscr{K}$.

If $A \in d^P\mathscr{K}$ then d_A has a canonical factorization

$$\begin{array}{ccccccccc} 0 & \longrightarrow & ZA & \xrightarrow{\zeta A} & A & \xrightarrow{\delta A} & BA & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle d_A} & & \downarrow & & \\ 0 & \longleftarrow & Z'A & \xleftarrow[\zeta' A]{} & A & \xleftarrow[\delta' A]{} & BA & \longleftarrow & 0 . \end{array} \tag{9.1}$$

Such a factorization being chosen for each $A \in d^P\mathscr{K}$, the rows of (9.1) are the values of functors $\mathbf{D}, \mathbf{D}' : d^P\mathscr{K} \longrightarrow \mathscr{K}^S$. Different choices would of course lead to naturally equivalent functors.

Then Z, B, Z' — the *cycle, boundary* and *dual-cycle* functors — are the compositions $S'\mathbf{D}$, $S''\mathbf{D} = S'\mathbf{D}'$, $S''\mathbf{D}'$, and $\zeta, \delta, \delta', \zeta'$, are the natural transformations $s'\mathbf{D}$, $s''\mathbf{D}$, $s'\mathbf{D}'$, $s''\mathbf{D}'$.

Since $d^2 = \delta'\delta\delta'\delta = 0$ it follows that $\delta\delta' = 0$ and thus that $\delta' = \zeta\beta$ for $\beta : BA \longrightarrow ZA$ a proper monomorphism and $\delta = \beta'\zeta'$ for $\beta' : Z'A \longrightarrow BA$ a proper epimorphism. This leads to two proper s.e.s.

$$\begin{array}{l} 0 \longrightarrow BA \xrightarrow{\beta A} ZA \xrightarrow{\eta A} HA \longrightarrow 0 \\ 0 \longrightarrow H'A \xrightarrow{\eta' A} Z'A \xrightarrow{\beta' A} BA \longrightarrow 0 \end{array} \tag{9.2}$$

which may again be considered as the values of functors $\mathbf{Z}, \mathbf{Z}' : d^P\mathscr{K} \rightarrow \mathscr{K}^S$. However the diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & BA & \longrightarrow & BA & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & ZA & \longrightarrow & A & \longrightarrow & BA & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H'A & \longrightarrow & Z'A & \longrightarrow & BA & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

obviously has proper exact rows and columns. Thus $\mathbf{Z}$ and $\mathbf{Z}'$ may be chosen so that $H = H'$. The common value $H = S''\mathbf{Z} = S'\mathbf{Z}'$ is of course the *homology* functor.

PROPOSITION 9.3. *$d^P\mathscr{K}$ is an additive category with cancellation.*

This is an immediate consequence of the fact that $\mathbf{D}$ and $\mathbf{D}'$ are additive functors, together with the fact that $\mathscr{K}$ and $\mathscr{K}^S$, as abelian categories, have cancellation.

It is not in general true that $d^P\mathscr{K}$ is an abelian category, at least with the class of proper maps inherited from $d\mathscr{K}$. However the notions of proper map and proper exact sequence in $d\mathscr{K}$ will be appropriated to $d^P\mathscr{K}$. A map in $d^P\mathscr{K}$ is proper if it is proper in $d\mathscr{K}$; a sequence in $d^P\mathscr{K}$ is proper exact if it is proper exact in $d\mathscr{K}$.

The proper s.e.s. of $d^P\mathscr{K}$ are the objects of a subcategory of $(d\mathscr{K})^S$ which will be denotedby $(d^P\mathscr{K})^S$. But while $(d\mathscr{K})^S$ and $d(\mathscr{K}^S)$ may obviously be identified the same is not true of $d^P(\mathscr{K}^S)$ and $(d^P\mathscr{K})^S$, the former being in general properly contained in the latter. For if $((0 \to A' \to A \to A'' \to 0), (d', d, d''))$ is in $d^P(\mathscr{K}^S)$ then $0 \to ZA' \to ZA \to ZA'' \to 0$, the kernel of (d', d, d''), is a proper s.e.s., which need not be the case in $(d^P\mathscr{K})^S$.

This observation is the source of the following example, which justifies the assertion that $d^P\mathscr{K}$ need not be an abelian category.

PROPOSITION 9.5. *If $d^P\mathscr{K}$ contains an object A such that $HA \neq 0$ then $d^P(\mathscr{K}^S)$ is not abelian.*

It will be shown that axiom (P 3) is violated. If (A, A, i, j, i', j') is a direct sum decomposition of $A + A$ in $\mathscr{K}$ the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D & \longrightarrow & C & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^X & \xrightarrow{\lambda} & A^X + A & \xrightarrow{\rho'} & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\theta} & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

in which A^X is of course $(A + A, i'j)$, $(A^X, A, \lambda, \rho, \lambda', \rho')$ is a direct sum decomposition of $A^X + A$ in $d\mathscr{K}$, $\varphi = j + dj'$ and $\theta = \varphi\rho + \rho'$, has clearly proper exact rows and columns.

But the third row is obviously an object of $d^P(\mathscr{K}^S)$; the second row is also in $d^P(\mathscr{K}^S)$, since the functor Z, applied to it, gives $0 \to A \to A + ZA \to 0$. If the diagram is regarded, reading downward, as a proper s.e.s. in $d\mathscr{K}^S$, the conclusion follows from the assertion that the first row is not in $d^P(\mathscr{K}^S)$.

But if Z is applied to the diagram it gives

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & ZD & \longrightarrow & ZC & \longrightarrow & ZA & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & A+ZA & \longrightarrow & ZA & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta\delta} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & ZA & \longrightarrow & ZA & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where, since $0 \to C \to A^X + A \to 0$ is split by $\lambda' : A \to A^X + A$, the second column is proper exact. If the top row were proper exact the first column would be too. But since $HA \neq 0$, β is not an epimorphism.

The functors Z, B, Z', and hence H, are exact on $d^P(\mathscr{K}^S)$. They are not, however, exact on all of $(d^P\mathscr{K})^S$, and this failure of exactness is measured by a certain natural transformation, which will now be studied.

LEMMA 9.6. *If $0 \to A' \to A \to A'' \to 0$ is a proper s.e.s. in $d^P\mathscr{K}$ then $0 \to ZA' \to ZA \to ZA''$ is proper exact. Dually, $Z'A' \to Z'A \to Z'A'' \to 0$ is proper exact.*

For $(\zeta A)(Za') = a'(\zeta A')$ is a proper monomorphism, hence so also is Za'. Dually Za'' is a proper epimorphism. But the columns and the two center rows in the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & ZA' & \longrightarrow & ZA & \longrightarrow & ZA'' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
 & & {\scriptstyle d'}\downarrow & & {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d''} & & \\
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & Z'A' & \longrightarrow & Z'A & \longrightarrow & Z'A'' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

are proper exact, so that the result follows from (4.3).

PROPOSITION 9.7. *If* $\mathbf{A} = (0 \to A' \to A \to A'' \to 0)$ *is a proper s.e.s. in* $d^P\mathscr{K}$ *there is a natural transformation* $\Delta\mathbf{A} : HA'' \to HA'$ *such that* $HA' \to HA \to HA'' \to HA' \to HA \to HA''$ *is proper exact.*

It is only necessary to consider the diagram

$$(9.8)\qquad \begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & HA' & \longrightarrow & HA & \longrightarrow & HA'' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & Z'A' & \longrightarrow & Z'A & \longrightarrow & Z'A'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & ZA' & \longrightarrow & ZA & \longrightarrow & ZA'' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & HA' & \longrightarrow & HA & \longrightarrow & HA'' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

which, by the preceding lemma, satisfies the hypothesis of (4.3).

Just as the notions of proper map and proper exact sequence were defined $d^P\mathscr{K}$ in terms of the category $d\mathscr{K}$, so the notion of a connected functor on $d^P\mathscr{K}$ is defined. (H, Δ) is of course an example of such a functor. The anticommutation relations of (6.6) of course hold for it.

The proper exact sequence of (9.7) may also be arranged in a triangle:

$$(9.9)\qquad \begin{array}{ccc} HA' & \longrightarrow & HA \\ & {\scriptstyle \Delta A}\nwarrow & \Big| \\ & & HA'' . \end{array}$$

A null-object, A^x say, of $d\mathscr{K}$ lies of course in $d^P\mathscr{K}$ and satisfies $H(A^x) = 0$. Thus by (8.2), H vanishes on null-homotopic maps and thus has the same value on homotopic maps. If $\mathfrak{f}$ is a class of homotopic maps in $d^P\mathscr{K}$ then $H\mathfrak{f}$ will be used for the common value of H on its elements.

In the graded case consideration will be restricted to categories admitting translation. If $\mathscr{K}$ is a graded abelian category admitting translation, $\partial^P\mathscr{K}$ may be defined in analogy with $d^P\mathscr{K}$ in the ungraded case. The factorization of (9.1), with d_A replaced by ∂_A, and the sequences (9.2), recur here, and because of the possibility of shifting degrees the convention may be made that all the maps which occur in (9.1) and (9.2) are of degree 0, with the exception of ∂, δ and β', which are of degree -1. The conclusions above all hold, with appropriate restrictions on degrees. In particular if $0 \longrightarrow A' \xrightarrow{a'} A \xrightarrow{a''} A'' \longrightarrow 0$ is a proper s.e.s. in $\partial^P\mathscr{K}$ and a', a'' are of degree 0 then Δ is of degree -1.

For any abelian category $\mathscr{K}$, then, $\partial^P(\mathscr{K}^\infty)$ is defined. For such a sequence as that just mentioned the triangle (9.9) becomes a proper exact sequence,

$$(9.10)\qquad \cdots \to H_{k+1}A'' \to H_kA' \to H_kA \to H_kA'' \to H_{k-1}A' \to \cdots .$$

10. The homology connecting homomorphism

It is useful to have a more explicit computation of the homology connecting homomorphism Δ than that given above. This is given by the following lemma.

LEMMA 10.1. *If* $\mathbf{A} = (0 \longrightarrow A' \xrightarrow{a'} A \xrightarrow{a''} A'' \longrightarrow 0)$ *is a proper s.e.s.* $d^P\mathscr{K}$, *if* $\varphi : X \to A$, $\varphi'' : X \to ZA''$, $\varphi' : X \to ZA'$ *and* $(\zeta A'')\varphi'' = a''\varphi$, $a'(\zeta A')\varphi' = d\varphi$ *then* $(\Delta\mathbf{A})(\eta A'')\varphi'' = (\eta A')\varphi'$.

This is just Lemma 4.4 applied to the maps $(\zeta' A)\varphi : X \to Z'A$, $(\eta' A'')(\eta A'')\varphi'' : X \to Z'A''$ and $\varphi' : X \to ZA'$ and the diagram (9.8).

A most important application of this lemma occurs when the sequence $\mathbf{A}$ splits as a sequence in $\mathscr{K}$. To study this case it is useful to introduce the covariant functorial involution Υ of $d\mathscr{K}$, defined by $\Upsilon(A, d) = (A, -d)$

on objects and $\Upsilon(A, A', f) = (\Upsilon A, \Upsilon A', f)$ on maps. Observe that this takes $d^P\mathscr{K}$ into itself. In defining the functors $\mathbf{D}, \mathbf{D}', \mathbf{Z}, \mathbf{Z}'$ it is no restriction to assume that

$$\begin{array}{lllll} Z\Upsilon = Z, & B\Upsilon = B, & Z'\Upsilon = Z', & H\Upsilon = H, & \\ \zeta\Upsilon = \zeta, & \delta\Upsilon = -\delta, & \delta'\Upsilon = \delta', & \zeta'\Upsilon = \zeta', & \beta\Upsilon = \beta, \\ \eta\Upsilon = \eta, & \eta'\Upsilon = \eta', & \beta'\Upsilon = -\beta'. & & \end{array}$$

Now suppose the proper s.e.s. $\mathbf{A}$ in $d\mathscr{K}$ splits in $\mathscr{K}$, i.e., there is a map $f : A'' \to A$ in $\mathscr{K}$ such that $a''f = 1 : A''$. Then $a''(df - fd'') = 0$ so that $df - fd'' = a'g$ for $g : A'' \to A'$ in $\mathscr{K}$. But

$$a'(d'g + gd'') = d(df - fd'') + (df - fd'')d'' = 0$$

so that $g : \Upsilon A'' \to A'$ in $d\mathscr{K}$. A change of the splitting map f results only in a nullhomotopic change in g. Thus for such a sequence $\mathbf{A}$ there is defined a homotopy class $\Delta^\#\mathbf{A}$ of maps $\Upsilon A'' \to A'$ in $d\mathscr{K}$.

LEMMA 10.2. *If the proper s.e.s.* $\mathbf{A}$ *in* $d^P\mathscr{K}$ *splits in* $\mathscr{K}$ *then* $\Delta\mathbf{A} = H\Delta^\#\mathbf{A}$.

This is a straightforward application of (10.1).

If $\mathscr{K}$ and $\mathscr{L}$ are abelian categories and $T : \mathscr{K} \to \mathscr{L}$ is an additive functor $d\mathscr{K} \to d\mathscr{L}$, which may without danger of confusion also be denoted by T, is defined by $T(A, d) = (TA, Td)$.

PROPOSITION 10.3. *Suppose* $\mathscr{K}$ *and* $\mathscr{L}$ *are abelian categories,* $T : \mathscr{K} \to \mathscr{L}$ *is an additive functor, and* $\mathbf{A}$ *is a proper s.e.s. in* $d\mathscr{K}$ *which splits in* $\mathscr{K}$. *Then* $T\mathbf{A}$ *is proper exact and splits in* $d\mathscr{K}$ *and* $\Delta^\# T\mathbf{A} \supset T\Delta^\#\mathbf{A}$. *Thus if* $T\mathbf{A}$ *is in* $d^P\mathscr{L}$ *then* $\Delta T\mathbf{A} = HT\Delta^\#\mathbf{A}$.

It is only necessary to observe that, in the covariant case for example, if $f : A'' \to A$ splits $\mathbf{A}$ then Tf splits $T\mathbf{A}$. If $\gamma : A'' \to A'$ satisfies $a'\gamma = df - fd''$ then $T\gamma$ satisfies $(Ta')(T\gamma) = (Td)(Tf) - (Tf)(Td'')$ so that $T\gamma$ is in $\Delta^\# T\mathbf{A}$.

If T is contravariant and $g : A \to A'$ satisfies $a'g + fa'' = 1 : A$ then Tg splits $T\mathbf{A}$. But $a'\gamma a'' = dfa'' - fd''a'' = d(1 : A - a'g) - (1 : A - a'g)d = a'(gd - d'g)$ so that $\gamma a'' = gd - d'g$ and $(Ta'')(T\gamma) = (Td)(Tg) - (Tg)(Td')$ and $T\gamma \in \Delta^\# T\mathbf{A}$.

If $\mathscr{K}$ is a graded abelian category admitting translation the homology connecting homomorphism in $\partial^P\mathscr{K}$ may be discussed in a similar manner. The functor Υ is unnecessary in this case. If

$$\mathbf{A} = (0 \longrightarrow A' \xrightarrow{a'} A \xrightarrow{a''} A'' \longrightarrow 0)$$

is a proper s.e.s. in $\partial\mathscr{K}$ with a', a'' of degree 0, say, and $\mathbf{A}$ is split in $\mathscr{K}$ by a map $f : A'' \to A$ of degree 0 then $\partial f - f\partial'' = a'g$ where $g : A'' \to A'$

is a map of degree -1 in $\partial\mathscr{K}$. A homotopy class $\Delta^{\#}\mathbf{A}$ of such maps is determined as in the ungraded case, and if $\mathbf{A}$ lies in $\partial_P\mathscr{K}$ the homology connecting homomorphism is $\Delta\mathbf{A} = H\Delta^{\#}\mathbf{A}$.

The effect of homogeneous functors is described by the analogue of (10.3).

11. Hom in $\partial\mathscr{K}$; cohomology

If $\mathscr{K}$ is a graded abelian category admitting translation the graded groups of homotopy classes of maps in $\partial\mathscr{K}$ may be regarded as homology groups by means of the following construction. Define the mixed functor $M:\partial\mathscr{K}\times\partial\mathscr{K}\to\partial\mathscr{G}^{\infty}$ by $M(X, Y) = \operatorname{Hom}(X, Y; \mathscr{K})$ with derivation $\partial\varphi = \partial_Y\varphi - (-1)^r\varphi\partial_X$ for φ homogeneous of degree r. Then $HM = \operatorname{Hom}: \partial\mathscr{K}\times\partial\mathscr{K}\to\mathscr{G}^{\infty}$.

Now suppose $\mathbf{W} = (0 \longrightarrow W' \xrightarrow{w'} W \xrightarrow{w''} W'' \longrightarrow 0)$ is a proper s.e.s. in $\partial\mathscr{K}$ with w' and w'' of degree 0. Then for $X \in \partial\mathscr{K}$ the short sequence

$$M(X, \mathbf{W}) = (0 \longrightarrow M(X, W') \longrightarrow M(X, W) \longrightarrow M(X, W'') \longrightarrow 0)$$

in $\partial\mathscr{G}^{\infty}$ is defined. If $M(X, \mathbf{W})$ is exact its exact homology sequence is

$$\begin{aligned} \cdots &\longrightarrow \operatorname{Hom}_{r+1}(X, W'') \xrightarrow{\Delta M(X,\mathbf{W})_{r+1}} \operatorname{Hom}_r(X, W') \\ &\longrightarrow \operatorname{Hom}_r(X, W) \longrightarrow \operatorname{Hom}_r(X, W'') \\ &\xrightarrow{\Delta M(X,\mathbf{W})_r} \operatorname{Hom}_{r-1}(X, W') \longrightarrow \cdots, \end{aligned} \tag{11.1}$$

in which the unlabelled maps are composition with w' and w'', or more properly, with their homotopy classes. Dually for $Y \in \partial\mathscr{K}$ the short sequence

$$M(\mathbf{W}, Y) = (0 \longrightarrow M(W'', Y) \longrightarrow M(W, Y) \longrightarrow M(W', Y) \longrightarrow 0)$$

is defined; if it is exact its exact homology sequence is

$$\begin{aligned} \cdots &\longrightarrow \operatorname{Hom}_{r+1}(W', Y) \xrightarrow{\Delta M(\mathbf{W}, Y)_{r+1}} \operatorname{Hom}_r(W'', Y) \\ &\longrightarrow \operatorname{Hom}_r(W, Y) \longrightarrow \operatorname{Hom}_r(W', Y) \\ &\xrightarrow{\Delta M(\mathbf{W}, Y)_r} \operatorname{Hom}_{r-1}(W'', Y) \longrightarrow \cdots, \end{aligned} \tag{11.2}$$

the unlabelled maps once more being composition with w' and w''.

If $\mathbf{X}$ and $\mathbf{Y}$ are proper s.e.s. in $\partial\mathscr{K}$ and all the short sequences $M(X'', \mathbf{Y}), \cdots, M(\mathbf{X}, Y'')$ are exact then (11.1) and (11.2) may be assembled in the diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathrm{Hom}_{r+1}(X', Y'') & \longrightarrow & \mathrm{Hom}_r(X', Y') & \longrightarrow & \mathrm{Hom}_r(X', Y) & \longrightarrow & \cdots \\
 & & \downarrow & * & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathrm{Hom}_r(X'', Y'') & \longrightarrow & \mathrm{Hom}_{r-1}(X'', Y') & \longrightarrow & \mathrm{Hom}_{r-1}(X'', Y) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathrm{Hom}_r(X, Y'') & \longrightarrow & \mathrm{Hom}_{r-1}(X, Y') & \longrightarrow & \mathrm{Hom}_{r-1}(X, Y) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

which commutes except in the starred squares where, according to (6.6), it must anticommute.

One case in which both $M(X, \mathbf{W})$ and $M(\mathbf{W}, Y)$ are both exact is that in which $\mathbf{W}$ splits in $\mathscr{K}$. In this case ΔM may be computed in the following way.

PROPOSITION 11.3. *If* $\mathbf{W}$ *is a proper s.e.s. in* $\partial\mathscr{K}$ *which splits in* $\mathscr{K}$ *and* $X, Y \in \partial\mathscr{K}$ *then* $\Delta M(X, \mathbf{W}) : \mathrm{Hom}(X, W'') \to \mathrm{Hom}(X, W')$ *is composition with* $\Delta^{\#}\mathbf{W}$, *while* $\Delta M(\mathbf{W}, Y)_r : \mathrm{Hom}_r(W', Y) \to \mathrm{Hom}_{r-1}(W'', Y)$ *is composition with* $(-1)^{r+1}\Delta^{\#}\mathbf{W}_r$.

This is implied by the assertion that $\Delta^{\#}M(X, \mathbf{W}) \supset M(X, \Delta^{\#}\mathbf{W})$, while $\Delta^{\#}M(\mathbf{W}, Y)_r \supset (-1)^{r+1}M(\Delta^{\#}\mathbf{W}, Y)_r$. To see this suppose that $\mathbf{W}$ is split by $f : W'' \to W$ and suppose also that $g : W \to W'$ satisfies $w'g + fw'' = 1 : W$. Then $\Delta^{\#}\mathbf{W}$ contains the map $\gamma : W'' \to W'$ of degree -1 such that $w'\gamma = \partial_W f - f\partial_{W''}$ and $\gamma w'' = g\partial_W - \partial_{W'}g$.

Now $\mathrm{Hom}(X, f ; \mathscr{K}) : M(X, W'') \to M(X, W)$ splits $M(X, \mathbf{W})$ in $\mathscr{G}^\infty$. Thus $\Delta^{\#}M(X, \mathbf{W})$ is represented by the map $\lambda : M(X, W'') \to M(X, W')$ such that for $\varphi : X \to W''$ of degree r, say, $w'(\lambda\varphi) = \partial(f\varphi) - f(\partial\varphi) = \partial_W(f\varphi) - (-1)^r(f\varphi)\partial_X - f[\partial_{W''}\varphi - (-1)^r\varphi\partial_X] = \partial_W f\varphi - f\partial_{W''}\varphi = w'\gamma\varphi = w'M(X, \gamma)\varphi$.

Similarly $\mathrm{Hom}(g, Y) : M(W', Y) \to M(W, Y)$ splits $M(\mathbf{W}, Y)$. Thus $\Delta^{\#}M(\mathbf{W}, Y)$ is represented by the map $\mu : M(W', Y) \to M(W, Y)$ such that for $\varphi : W' \to Y$ of degree r, say,

$$\begin{aligned}(\mu\varphi)w'' &= \partial(\varphi g) - (\partial\varphi)g = \partial_Y\varphi g - (-1)^r\varphi g\partial_W - [\partial_Y\varphi - (-1)^r\varphi\partial_{W'}]g \\ &= (-1)^{r+1}[\varphi g\partial_W - \varphi\partial_{W'}g] = (-1)^{r+1}[M(\gamma, Y)\varphi]w'' .\end{aligned}$$

$\mathrm{Hom}(X, Y)$ deserves special attention in the case that either X or Y has derivation zero. Such objects may be identified with objects of $\mathscr{K}$;

properly speaking, a functorial injection $\mathscr{K} \to d\mathscr{K}$ is defined by $A \to (A, 0)$. No notation will be introduced for this ; rather, Hom (X, Y) will also be written for X or Y in $\mathscr{K}$, with the injection understood.

If $X \in \mathscr{K}$, $Y \in \partial\mathscr{K}$ then Hom (X, Y) is the *left cohomology of Y with coefficients X* denoted by $H_*(X; Y)$, with homogeneous components $H_r(X; Y) = \mathrm{Hom}_r(X, Y)$. Dually if $X \in \partial\mathscr{K}$, $Y \in \mathscr{K}$ then Hom (X, Y) is the *right cohomology of X with coefficients* Y, denoted by $H^*(X; Y)$, with homogeneous components $H^r(X; Y) = \mathrm{Hom}_{-r}(X, Y)$.

Right cohomology is of course what is usually called cohomology. It should be noted that it is dual not to homology but to left cohomology. Left cohomology is traditionally neglected.

Observe that under suitable conditions (11.1) and (11.2) are exact sequences of cohomology groups. If $\mathbf{W}$ is a proper s.e.s. in $\partial\mathscr{K}$, X or Y is in $\mathscr{K}$ and $M(\mathbf{X}, W)$ is exact, or $M(\mathbf{W}, Y)$ is exact then (11.1) or (11.2) is the *cohomology sequence* of $\mathbf{W}$ with coefficients X or Y. If X or $Y \in \partial\mathscr{K}$ and $\mathbf{W}$ is a proper s.e.s in $\mathscr{K}$ then, if $M(X, \mathbf{W})$ or $M(\mathbf{W}, Y)$ is exact, (11.1) or (11.2) is the *coefficient cohomology sequence* of X or Y with coefficients $\mathbf{W}$.

For cohomology the following additional notation is introduced :

cocycle group	$Z_r(X; Y) = Z_rM(X, Y)$	$Z^r(X; Y) = Z_{-r}M(X, Y)$
coboundary group	$B_r(X; Y) = B_rM(X, Y)$	$B^r(X; Y) = B_{-r}M(X, Y)$
dual cocycle group	$Z'_r(X; Y) = Z'_rM(X, Y)$	$Z'^r(X; Y) = Z'_{-r}M(X, Y)$.

If both X and Y are in $\mathscr{K}$, i.e., both X and Y have derivation zero then so has $M(X, Y)$ and Hom (X, Y) is just Hom $(X, Y; \mathscr{K})$.

Hom in the ungraded case may also be treated in the above fashion, with similar definitions for cohomology. This discussion will be omitted here.

The following considerations are restricted to associated graded categories of abelian categories. If $\mathscr{K}$ is an abelian category then cohomology in $\partial\mathscr{K}^\infty$ may also be defined for coefficients in $\mathscr{K}$, an object of $\mathscr{K}$ being identified with the object of $\mathscr{K}^\infty$ which is equal to it in degree zero and is zero in other degrees. In fact it is sufficient to consider only such coefficients : the groups with coefficients in $\mathscr{K}^\infty$ are of course given by

$$H_r(X; Y) \approx \Pi_k H_{r-k}(X_k; Y) , \quad H^r(X; Y) \approx \Pi_k H^{r+k}(X; Y_k) ,$$

the equivalences being natural.

An object X of $\mathscr{K}^\infty$, or by extension of $\partial\mathscr{K}^\infty$, is *positive* if $X_k = 0$ for $k < 0$. If $X \in \partial^p\mathscr{K}^\infty$ is positive then $Z_0X = X_0$ and $(\eta X)_0 : X \to H_0X$ is a proper epimorphism in $\partial\mathscr{K}^\infty$. If $0 \to X' \to X \to H_0X \to 0$ is proper exact in $\partial\mathscr{K}^\infty$ then X' is also in $\partial^P\mathscr{K}^\infty$, with $H_0X' = 0$ and $H_kX' \approx H_kX$ for $k \neq 0$. The cohomology connecting homomorphism of this sequence with arbitrary coefficients $C \in \mathscr{K}$ need not of course exist, but the start at least of the right cohomology sequence, i.e.,

$$0 \longrightarrow H^0(H_0X\,;C) \longrightarrow H^0(X\,;C) \longrightarrow H^0(X'\,;C)$$

is exact. But the last of these groups is clearly trivial. This gives the following result.

LEMMA 11.4. *If* $X \in \partial^P\mathscr{K}^\infty$ *is positive and* $C \in \mathscr{K}$ *then* $H^0(X\,;C) \approx \operatorname{Hom}(H_0X, C\,;\mathscr{K})$.

The following lemma is proved by recursion in standard fashion.

LEMMA 11.5. *If* $X \in \partial\mathscr{K}^\infty$ *is positive and projective in* $\mathscr{K}^\infty$, $Y \in \partial^P\mathscr{K}^\infty$ *and* $HY = 0$ *then* $\operatorname{Hom}(X, Y) = 0$.

PROPOSITION 11.6. *If* $X \in \partial\mathscr{K}^\infty$ *is positive and projective in* $\mathscr{K}^\infty$ *and* $Y \in \partial^P\mathscr{K}^\infty$ *has* $H_kY = 0$ *for* $k \neq 0$ *then* $\operatorname{Hom}(X, Y) \approx H^*(X\,;H_0Y)$. *In particular, if* $X \in \partial^P\mathscr{K}^\infty$ *then* $\operatorname{Hom}_0(X, Y) \approx \operatorname{Hom}(H_0X, H_0Y\,;\mathscr{K})$.

Let Y' be defined by $Y'_k = Y_k$ for $k > 0$, $Y'_0 = Z_0Y$ and $Y'_k = 0$ for $k < 0$. Then $0 \to Y' \to Y \to Y'' \to 0$ is a proper s.e.s. in $\partial^P\mathscr{K}^\infty$ with $HY'' = 0$.

But also $0 \to W \to Y' \to H_0Y \to 0$ is a proper s.e.s. in $\partial^P\mathscr{K}^\infty$ with $HW = 0$. The result now follows from the two preceding lemmas and the exactness of (11.1).

The duals of the last few results are not stated; they may easily be supplied by the reader.

12. The extended category

If $\mathscr{K}$ is an abelian category with enough projectives then for $A, B \in \mathscr{K}$ the groups $\operatorname{Ext}^r(A, B)$ may be defined, as in [2], up to a transitive family of isomorphisms. A somewhat different procedure will be followed here. Under these assumptions the following additional structure may be imposed — in a non-unique fashion to be sure — on $\mathscr{K}$. For every $A \in \mathscr{K}$ let $\mathbf{P}A$ be a proper s.e.s. such that $S''\mathbf{P}A = A$ and $S\mathbf{P}A$ is projective. Such a system of choices will be referred to as an *assignment of projectives.* The notation $S\mathbf{P}A = PA$, $S'\mathbf{P}A = \Omega A$, $s''\mathbf{P}A = \pi A$, $s'\mathbf{P}A = \omega A$ will be used, so that $\mathbf{P}A$ is the sequence

$$0 \longrightarrow \Omega A \xrightarrow{\omega A} PA \xrightarrow{\pi A} A \longrightarrow 0\,.$$

Then for each $A \in \mathscr{K}$ an object $\mathbf{PR}A \in \partial^P\mathscr{K}^\infty$, the *projective resolution* of A, is defined by

$$(\mathbf{PR}A)_k = \begin{cases} P\Omega^k A & k \geqq 0 \\ 0 & k < 0 , \end{cases}$$

the derivation being given by

$$(\omega\Omega^{k-1}A)(\pi\Omega^k A) : (\mathbf{PR}A)_k \longrightarrow (\mathbf{PR}A)_{k-1} .$$

Here $\Omega^0 A = A$ and $\Omega^k A = \Omega(\Omega^{k-1}A)$.

Finally, relative to such an assignment of projectives, a graded additive category $\mathrm{Ext}\mathscr{K}$, the *projective extended category of* $\mathscr{K}$, is defined. Its objects are the objects of $\mathscr{K}$, while

$$\mathrm{Hom}\,(A, B\,;\, \mathrm{Ext}\mathscr{K}) = \mathfrak{Hom}\,(\mathbf{PR}A, \mathbf{PR}B) .$$

Now $\pi A : (\mathbf{PR}A)_0 \to A$ gives rise to an isomorphism $\sigma_A : H_0(\mathbf{PR}A) \approx A$, and of course $H_k\mathbf{PR}A = 0$ for $k > 0$. By (11.6) then $\mathfrak{Hom}_0\,(\mathbf{PR}A, \mathbf{PR}B) = \mathrm{Hom}_0\,(A, B\,;\, \mathrm{Ext}\mathscr{K})$ is naturally isomorphic to $\mathrm{Hom}\,(A, B\,;\, \mathscr{K})$. Thus $\mathscr{K}$ may be considered to be a subcategory of $\mathrm{Ext}\,\mathscr{K}$, in fact that subcategory which consists of the maps of degree zero.

If another assignment of projectives, say $\mathbf{P}'$, is made, Proposition 11.6 defines a unique isomorphism of the category $\mathrm{Ext}\,\mathscr{K}$ with the corresponding category $\mathrm{Ext}'\mathscr{K}$ defined relative to $\mathbf{P}'$. That is, the category $\mathrm{Ext}\mathscr{K}$ is unique up to a transitive family of isomorphisms.

It should be remarked that the category $\mathrm{Ext}\mathscr{K}$ does not admit translation. In fact for any $A, B \in \mathscr{K}$ the groups $\mathrm{Hom}_r\,(A, B\,;\, \mathrm{Ext}\mathscr{K})$ are zero for r positive. For $r \leqq 0$ they are of course the values of the functor Ext:

$$\mathrm{Ext}^r\,(A, B\,;\, \mathscr{K}) = \mathrm{Hom}_{-r}\,(A, B\,;\, \mathrm{Ext}\,\mathscr{K}) .$$

It will in general be assumed below that, in any appropriate category, an assignment of projectives has been made, so that the extended category is defined.

If $\mathscr{K}$ is an abelian category with enough projectives then so also is $\mathscr{K}^S$. If $\mathbf{A} = (0 \longrightarrow A' \xrightarrow{a'} A \xrightarrow{a''} A'' \longrightarrow 0)$ is a proper s.e.s. in $\mathscr{K}$ then $S'\mathbf{PRA}$, $S\mathbf{PRA}$, $S''\mathbf{PRA}$ are in $\partial^P\mathscr{K}^\infty$ and by (11.6) there are homotopy equivalences $S'\mathbf{PRA} \to \mathbf{PR}A'$, $S\mathbf{PRA} \to \mathbf{PR}A$, $S''\mathbf{PRA} \to \mathbf{PR}A''$, which are unique up to homotopy, such that the diagram

$$\begin{array}{ccccc} S'\mathbf{PRA} & \xrightarrow{s'\mathbf{PRA}} & S\mathbf{PRA} & \xrightarrow{s''\mathbf{PRA}} & S''\mathbf{PRA} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{PR}A' & \xrightarrow{a'} & \mathbf{PR}A & \xrightarrow{a''} & \mathbf{PR}A'' \end{array}$$

commutes up to homotopy.

Now **PRA** is in $(\partial^P \mathscr{K}^\infty)^S$ and its terms are projective in $\mathscr{K}^\infty$. Thus it splits in $\mathscr{K}^\infty$ and defines a homotopy class $\Delta^\#\mathbf{PRA}$ of maps $S''\mathbf{PRA} \to S'\mathbf{PRA}$ of degree -1. This may be composed with the above equivalences to give a homotopy class $\Delta\mathbf{A}$ of maps of $\mathbf{PR}A''$ into $\mathbf{PR}A'$, i.e., an element of $\mathrm{Ext}^1(A'', A'; \mathscr{K}) = \mathrm{Hom}_{-1}(A'', A'; \mathrm{Ext}\mathscr{K})$. But this is a connecting homomorphism for the injection $\mathscr{K} \to \mathrm{Ext}\mathscr{K}$, so that the injection is a connected functor. If also $C \in \mathscr{K}$ then the sequences (11.1) and (11.2) may be written for **PA** and C; together with (11.3) then give the following result.

PROPOSITION 12.1. *If* **A** *is a proper s.e.s. in* $\mathscr{K}$ *and* $C \in \mathscr{K}$ *then the sequences*

$$\cdots \longrightarrow \mathrm{Ext}^{r-1}(C, A'') \longrightarrow \mathrm{Ext}^r(C, A') \longrightarrow \mathrm{Ext}^r(C, A) \longrightarrow \mathrm{Ext}^r(C, A'') \longrightarrow \mathrm{Ext}^{r+1}(C, A') \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \mathrm{Ext}^{r-1}(A', C) \longrightarrow \mathrm{Ext}^r(A'', C) \longrightarrow \mathrm{Ext}^r(A, C) \longrightarrow \mathrm{Ext}^r(A', C) \longrightarrow \mathrm{Ext}^{r+1}(A'', C) \longrightarrow \cdots$$

are exact, the maps being composition in $\mathrm{Ext}\mathscr{K}$ *with* a', a'' *and* $\Delta\mathbf{A}$.

If $\mathscr{K}$ is, instead, a category with enough injectives, an *assignment of injectives* may be made in $\mathscr{K}$, the notion being dual to that of an assignment of projectives. Proceeding in dual fashion, the *injective resolution* $\mathbf{PR}^*A$ of an $A \in \mathscr{K}$ is defined, and the *injective extended category* $\mathrm{Ext}^*\mathscr{K}$ of $\mathscr{K}$ is constructed, where $\mathrm{Hom}(A, B; \mathrm{Ext}^*\mathscr{K}) = \mathrm{Hom}(\mathbf{PR}^*A, \mathbf{PR}^*B)$. As in the projective case, $\mathscr{K}$ may be identified with the subcategory of $\mathrm{Ext}^*\mathscr{K}$ consisting of the maps of degree zero.

PROPOSITION 12.2. *If* $\mathscr{K}$ *has both enough projectives and enough injectives then* $\mathrm{Ext}\mathscr{K}$ *and* $\mathrm{Ext}^*\mathscr{K}$ *are isomorphic.*

For by (11.6) and its dual, $\mathrm{Hom}(\mathbf{PR}A, \mathbf{PR}B) \approx \mathrm{Hom}(\mathbf{PR}A, B) \approx \mathrm{Hom}(\mathbf{PR}A, \mathbf{PR}^*B) \approx \mathrm{Hom}(A, \mathbf{PR}^*B) \approx \mathrm{Hom}(\mathbf{PR}^*A, \mathbf{PR}^*B)$.

In spite of this isomorphism, it seems preferable not to identify these categories.

13. Derived functors

If $\mathscr{K}$ and $\mathscr{L}$ are abelian categories, $\mathscr{K}$ has enough projectives and $F: \mathscr{K} \to \mathscr{L}$ is a proper covariant additive functor then, relative to an assignment of projectives in $\mathscr{K}$, the *left derived* functor $LF: \mathrm{Ext}\mathscr{K} \to \mathscr{L}^\infty$ is defined as follows. If $A \in \mathscr{K}$ then $F\mathbf{PR}A \in \partial^P \mathscr{L}^\infty$ and if $\varphi \in \mathrm{Ext}^r(A, B; \mathscr{K})$ then $F\varphi$ is a class of homotopic maps $F\mathbf{PR}A \to F\mathbf{PR}B$ of degree $-r$. Then $(LF)A = HF\,\mathbf{PR}A$ and $(LF)\varphi = HF\varphi$. Thus

LF is a homogeneous functor. Clearly $(L_kF)A = 0$ for $k < 0$.

If F is contravariant the contravariant homogeneous functor $RF: \mathrm{Ext}\mathscr{K} \to \mathscr{L}^\infty$, the *right derived* functor of F, is defined in analogous fashion. Its homogeneous components are written $(R^kF)A = H_{-k}F\mathbf{PR}A$, and satisfy $(R^kF) = 0$ for $k < 0$.

Dually if $\mathscr{K}$ has enough injectives the *right derived* functor $RF: \mathrm{Ext}^*\mathscr{K} \to \mathscr{L}^\infty$ of a covariant $F: \mathscr{K} \to \mathscr{L}$ and the *left derived* functor $LF: \mathrm{Ext}^* \mathscr{K} \to \mathscr{L}^\infty$ of a contravariant $F: \mathscr{K} \to \mathscr{L}$ are defined.

Though these definitions are relative to an assignment of projectives or injectives the dependence is only up to natural equivalence. For instance, in the first case considered, if $\mathbf{P}$ and $\mathbf{P}'$ are two assignments of projectives in $\mathscr{K}$, with associated extension categories $\mathrm{Ext}\mathscr{K}$ and $\mathrm{Ext}'\mathscr{K}$ derived functors LF and $L'F$ then the diagram

$$\begin{array}{ccc} \mathrm{Ext}\mathscr{K} & \xrightarrow{LF} & \mathscr{L}^\infty \\ {\scriptstyle T}\downarrow & & \uparrow \\ \mathrm{Ext}'\mathscr{K} & \xrightarrow{L'F} & \mathscr{L}^\infty \end{array}$$

commutes up to natural equivalence, where T is the isomorphism of § 12.

Now since the injections $\mathscr{K} \to \mathrm{Ext}\,\mathscr{K}$ and $\mathscr{K} \to \mathrm{Ext}^*\mathscr{K}$ are connected, derived functors also become connected. For example if $F: \mathscr{K} \to \mathscr{L}$ is proper covariant and $\mathbf{A} = (0 \longrightarrow A' \xrightarrow{a'} A \xrightarrow{a''} A'' \longrightarrow 0)$ is a proper s.e.s. in $\mathscr{K}$ then, provided $\mathscr{K}$ has enough projectives, $(LF)(\Delta\mathbf{A}): A'' \to (LF)A'$, of degree -1, in $\mathscr{L}^\infty$, giving rise to a sequence

$$(13.1) \qquad \cdots \to (L_{k+1}F)A'' \to (L_kF)A' \to (L_kF)A \to (L_kF)A'' \to \cdots .$$

As in the definition of $\Delta\mathbf{A}$, the terms of this sequence may be computed by means of a projective resolution $\mathbf{PR}A$ in $(\partial^P\mathscr{K}^\infty)^S$ which of course splits in $\mathscr{K}^\infty$ so that $F\mathbf{PR}A$ is exact, and in fact splits, in $\mathscr{L}^\infty$. By (10.2), or rather its analogue for the graded case, (13.1) is, up to equivalences, just the homology sequence of $F\mathbf{PR}A$.

PROPOSITION 13.2. *The sequence* (13.1), *as well as the corresponding sequences for LF and RF in both the covariant and contravariant cases, are exact.*

A case of especial interest is that in which a covariant functor $F: \mathscr{K} \to \mathscr{L}$ is *right-exact*, i.e., if whenever $0 \to A' \to A \to A'' \to 0$ is a proper s.e.s. in $\mathscr{K}$ then $FA' \to FA \to FA'' \to 0$ is proper exact. Such a functor is always proper, as may be seen by applying it to the canonical factorization of a proper map.

PROPOSITION 13.3. *If* $F: \mathscr{K} \to \mathscr{L}$ *is covariant right-exact then* L_0F *is naturally equivalent to* F.

For if $A \in \mathscr{K}$ then $(\mathbf{PR}A)_1 \to (\mathbf{PR}A)_0 \to A \to 0$ is proper exact, hence also $F(\mathbf{PR}A)_1 \to F(\mathbf{PR}A)_0 \to FA \to 0$.

This result has of course several duals, covariant being replaced by contravariant and left-derived being replaced by right-derived-functors, and right-exactness by left-exactness in appropriate combinations, the last notion being dual to that of right-exactness.

The functor Hom, considered as a functor in either one of its variables, is left-exact. Its right derived functor with respect to either variable is easily seen to be naturally equivalent to Ext.

It is useful for computational purposes to extend the notion of projective resolution in the following fashion. If $\mathscr{K}$ is abelian with enough projectives a *projective resolution* of $A \in \mathscr{K}$ is a pair (X, ξ) where $X \in \partial^P \mathscr{K}^\infty$ is positive and projective in $\mathscr{K}^\infty$, $H_k X = 0$ for $k > 0$ and $\xi : H_0 X \approx A$. In particular then for an assignment of projectives in $\mathscr{K}$, $(\mathbf{PR}A, \sigma_A)$ is a projective resolution of A.

If (X, ξ) is a projective resolution of A there is by (11.6) a unique homotopy class of homotopy equivalences $f : X \to \mathbf{PR}A$ such that $\xi(Hf) = \sigma_A$. For any proper covariant F then, $Hf : HFX \approx (LF)A$ is a uniquely determined isomorphism. Similar conclusions hold of course for contravariant functors, and *injective resolutions*, defined dually. Thus for computation of derived functors the projective resolution associated with an assignment of projectives may be replaced by an arbitrary projective resolution, the same statement holding also for injective resolutions.

For example if (X, ξ), (Y, η) are projective resolutions of A, B then $\mathfrak{Hom}\,(X, Y)$ is canonically isomorphic to Ext (A, B).

14. Ext in the category $\mathscr{K}^S$

If $\mathscr{K}$ is an abelian category with enough projectives then so also is $\mathscr{K}^S$. Since the structure of $\mathscr{K}$ determines that of $\mathscr{K}^S$ it seems plausible that Ext in the category $\mathscr{K}^S$ should be related to Ext in $\mathscr{K}$. Such a relation will be exhibited here.

For the purpose of this computation the following construction will first be introduced. If $\mathbf{A} = (0 \longrightarrow A' \xrightarrow{a'} A \xrightarrow{a''} A'' \longrightarrow 0)$ is a proper s.e.s. in an abelian category $\mathscr{L}$ let $\mathbf{A}'$, $\mathbf{A}''$ be the proper s.e.s. $(0 \to A' \to A' \to 0 \to 0)$ and $(0 \to 0 \to A'' \to A'' \to 0)$ and set $\mathbf{a}' = (1 : A', a', 0) : \mathbf{A}' \to \mathbf{A}$ and $\mathbf{a}'' = (0, a', 1 : A'') : \mathbf{A} \to \mathbf{A}''$ so that $0 \to \mathbf{A}' \to \mathbf{A} \to \mathbf{A}'' \to 0$ is a proper s.e.s. $\hat{\mathbf{A}}$ in $\mathscr{L}^S$. Then $\hat{\mathbf{A}}$ is the value on

A of a functor $\mathscr{L}^S \to \mathscr{L}^{SS}$, and $\mathbf{A}' = S'\hat{\mathbf{A}}$, $\mathbf{A}'' = S''\hat{\mathbf{A}}$ of functors $\mathscr{L}^S \to \mathscr{L}^S$, while $\mathbf{a}' = s'\hat{\mathbf{A}}$ and $\mathbf{a}'' = s''\hat{\mathbf{A}}$ are natural transformations.

LEMMA 14.1. *If* **A** *and* $\mathbf{B} = (0 \to B' \to B \to B' \to 0)$ *are proper s.e.s. in* $\mathscr{L}$ *then*

$$\begin{aligned}
\mathrm{Hom}\,(\mathbf{A}', \mathbf{B}'\ ; \mathscr{L}^S) &\approx \mathrm{Hom}\,(A', B'\ ; \mathscr{L})\\
\mathrm{Hom}\,(\mathbf{A}', \mathbf{B}\ ; \mathscr{L}^S) &\approx \mathrm{Hom}\,(A', B'\ ; \mathscr{L})\\
\mathrm{Hom}\,(\mathbf{A}', \mathbf{B}''; \mathscr{L}^S) &= 0\\
\mathrm{Hom}\,(\mathbf{A}\ , \mathbf{B}'\ ; \mathscr{L}^S) &\approx \mathrm{Hom}\,(A\ , B'\ ; \mathscr{L})\\
\mathrm{Hom}\,(\mathbf{A}\ , \mathbf{B}''; \mathscr{L}^S) &\approx \mathrm{Hom}\,(A'', B''; \mathscr{L})\\
\mathrm{Hom}\,(\mathbf{A}'', \mathbf{B}'\ ; \mathscr{L}^S) &\approx \mathrm{Hom}\,(A'', B'\ ; \mathscr{L})\\
\mathrm{Hom}\,(\mathbf{A}'', \mathbf{B}\ ; \mathscr{L}^S) &\approx \mathrm{Hom}\,(A'', B\ ; \mathscr{L})\\
\mathrm{Hom}\,(\mathbf{A}'', \mathbf{B}''; \mathscr{L}^S) &\approx \mathrm{Hom}\,(A'', B''; \mathscr{L})
\end{aligned}$$

are all natural equivalences.

Now suppose **A** and **B** are proper s.e.s. in the abelian category $\mathscr{K}$, which has enough projectives. Let $\mathbf{X} = (0 \to X' \to X \to X'' \to 0) = \mathbf{PRA}$ and $\mathbf{Y} = (0 \to Y' \to Y \to Y'' \to 0) = \mathbf{PRB}$. Then $(\hat{\mathbf{X}}, \hat{\sigma}_A)$ is a projective resolution of $\hat{\mathbf{A}}$ and $(\hat{\mathbf{Y}}, \hat{\sigma}_B)$ is a projective resolution of $\hat{\mathbf{B}}$, so that $(X', S'\sigma_A), \cdots$ are projective resolutions of $A', \cdots$.

The functor $M : \partial\mathscr{K}^\infty \times \partial\mathscr{K}^\infty \to \partial\mathscr{G}^\infty$ is defined in § 11 ; $M(X, Y)$ is $\mathrm{Hom}(X, Y; \mathscr{K}^\infty)$ with a derivation deduced from those of X and Y. The corresponding functor for s.e.s. in $\mathscr{K}$ will be denoted by $\mathbf{M} : \partial(\mathscr{K}^S)^\infty \times \partial(\mathscr{K}^S)^\infty \to \partial\mathscr{G}^\infty$.

Now if **M** is applied to the proper s.e.s. $\hat{\mathbf{X}}$ and $\hat{\mathbf{Y}}$ in $\partial(\mathscr{K}^S)^\infty$ it gives the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{M}(\mathbf{X}'', \mathbf{Y}') & \longrightarrow & \mathbf{M}(\mathbf{X}, \mathbf{Y}') & \longrightarrow & \mathbf{M}(\mathbf{X}', \mathbf{Y}') & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{M}(\mathbf{X}'', \mathbf{Y}) & \longrightarrow & \mathbf{M}(\mathbf{X}, \mathbf{Y}) & \longrightarrow & \mathbf{M}(\mathbf{X}', \mathbf{Y}) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{M}(\mathbf{X}'', \mathbf{Y}'') & \longrightarrow & \mathbf{M}(\mathbf{X}, \mathbf{Y}'') & \longrightarrow & \mathbf{M}(\mathbf{X}', \mathbf{Y}'') & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

which commutes and has exact rows and columns. This may be evaluated by (14.1) as follows :

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M(X'', Y') & \longrightarrow & M(X, Y') & \longrightarrow & M(X', Y') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M(X'', Y) & \longrightarrow & \mathbf{M}(\mathbf{X}, \mathbf{Y}) & \longrightarrow & M(X', Y') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M(X'', Y'') & \longrightarrow & M(X'', Y'') & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\tag{14.2}
$$

again commutes and has exact rows and columns.

By (4.2), then, the sequence

$$\mathbf{Q} = (0 \to M(X'', Y') \to \mathbf{M}(\mathbf{X}, \mathbf{Y}) \to M(X', Y') \dot{+} M(X'', Y'') \to 0)$$

is exact, where "$\dot{+}$" denotes the direct sum in $\mathscr{G}$ or $\mathscr{G}^{\infty}$.

The connecting homotopy classes for the first row and the first column of (14.2) are computed in Proposition 11.3. By naturality, they determine the connecting homotopy class of $\mathbf{Q}$. The following result comes, then, by applying the homology functor to $\mathbf{Q}$.

THEOREM 14.3. *If $\mathscr{K}$ is an abelian category with enough projectives and* $\mathbf{A} = (0 \to A' \to A \to A'' \to 0)$ *and* $\mathbf{B} = (0 \to B' \to B \to B'' \to 0)$ *are proper s.e.s. in $\mathscr{K}$ then the sequence*

$$\cdots \longrightarrow \operatorname{Ext}^r(A'', B' ; \mathscr{K}) \longrightarrow \operatorname{Ext}^r(\mathbf{A}, \mathbf{B} ; \mathscr{K}^S) \longrightarrow$$

$$\longrightarrow \operatorname{Ext}^r(A', B' ; \mathscr{K}) \dot{+} \operatorname{Ext}^r(A'', B'' ; \mathscr{K}) \xrightarrow{\nabla} \operatorname{Ext}^{r+1}(A'', B' ; \mathscr{K}) \longrightarrow \cdots$$

is exact, where ∇ is given by $\nabla(\varphi' \dot{+} \varphi'') = (\Delta\mathbf{B})\varphi'' + (-1)^{r+1}\varphi'(\Delta\mathbf{A})$.

If $\mathscr{K}$ has, instead, enough injectives, then a similar result holds with Ext replaced by Ext*.

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