HIGHER K-THEORY FOR CATEGORIES WITH EXACT SEQUENCES

DANIEL QUILLEN

To a ring A with identity is attached a sequence of abelian groups K_iA, $i \geq 0$ which may be defined as follows. Let $\mathcal{P}_A$ be the category of finitely generated left A-modules, endowed with the direct sum operation. By work of Segal and Anderson (cf. [1]), a category with a coherent associative and commutative operation such as $\mathcal{P}_A$ determines a connected generalized cohomology theory. The groups K_iA are the coefficient groups of this cohomology theory. One can prove that they agree with the K-groups in degrees ≤ 2 introduced by Bass and Milnor (cf. [4]), and with the ones computed for a finite field in [5].

However, it is clear from the existing K-theory in low degrees that, in order to establish the basic properties of K_*A for regular rings A, one requires K-groups for the category of all finitely generated A-modules, in which the relations come from exact sequences, not just direct sums. In the present paper we outline a higher K-theory for categories with exact sequences, which enables one to prove the homotopy axiom: $K_*A = K_*(A[T])$ for regular rings, and a localization exact sequence for Dedekind domains. Full details will appear elsewhere.

§1. The space $BGL(A)^+$ **and the groups** K_iA. Let $f : X \to Y$ be a map of connected CW complexes with basepoint. We call f <u>acyclic</u> if the following equivalent conditions are satisfied:

(i) $H_*(X, f^*L) \xrightarrow{\sim} H_*(Y, L)$ for any local coefficient system L on Y.

(ii) The homotopy-theoretic fibre F of f is an acyclic space, i.e. $\tilde{H}_*(F, Z) = 0$. (F is the space of pairs (x, p), where $x \in X$ and p is a path joining f(x) to the basepoint of Y.)

If f is acyclic, then $\pi_1(X)/N \xrightarrow{\sim} \pi_1(Y)$, where N is a normal subgroup of $\pi_1(X)$ which is perfect (equal to its commutator subgroup). Conversely, given a connected CW complex X and a perfect normal subgroup N of its fundamental group, one shows there exists an acyclic map f with source X, which is unique up to homotopy, such that N is the kernel of $\pi_1(f)$.

Now let A be a ring (supposed always to be associative with identity), let GL(A) be its infinite general linear group, and let BGL(A) be a classifying space for the discrete group GL(A). The commutator subgroup E(A) of $GL(A) = \pi_1(BGL(A))$ is perfect, so by the preceding there exists an acylic map

$$f : BGL(A) \to BGL(A)^+$$

unique up to homotopy, such that E(A) is the kernel of $\pi_1(f)$. The K-groups of the ring A are defined to be the homotopy groups of the space $BGL(A)^+$:

$$K_iA = \pi_i(BGL(A)^+) \quad \text{for} \quad i \geq 1.$$

These groups are closely connected with the homology of GL(A) and related groups, such as E(A) and the Steinberg group St(A). One has isomorphisms

$$K_1A = H_1(GL(A), \mathbf{Z})$$
$$K_2A = H_2(E(A), \mathbf{Z})$$
$$K_3A = H_3(St(A), \mathbf{Z})$$

showing that the above definition agrees with the K_1 of Bass and the K_2 of Milnor. Moreover, $BGL(A)^+$ is a loop space, which has the same homology as BGL(A) as f is acyclic. Thus by a theorem of Milnor and Moore one has isomorphisms

$$K_iA \otimes \mathbf{Q} \xrightarrow{\sim} \mathcal{P}H_i(GL(A), \mathbf{Q})$$

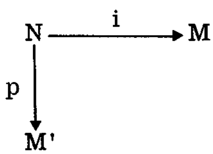
where $\mathcal{P}$ denotes the subspace of primitive elements.

There are two basic examples where the K-groups have been calculated in all dimensions. The case of a finite field is treated in [5]. When A is the ring of S-integers in a number field, Borel [2] has determined the groups $K_*A \otimes \mathbf{Q}$. In both cases one proceeds by computing the homology of GL(A) with appropriate coefficients, using techniques special to the type of ring under consideration.

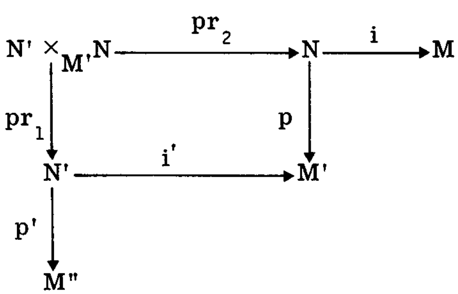
Starting from these examples, the theorems that follow may be used to produce many rings A for which the K-groups, or at least the groups $K_*A \otimes \mathbf{Q}$, can be determined.

§2. Higher K-groups for categories with exact sequences. Let $\mathcal{A}$ be a small abelian category, and let $\mathfrak{M}$ denote a full subcategory of $\mathcal{A}$ closed under extensions and containing the zero object. If M is an object of $\mathfrak{M}$, then by an $\mathfrak{M}$-subquotient of M, we mean a quotient of the form M_2/M_1, where M_1 and M_2 are subobjects of M such that $M_1 \subset M_2$, and such that M_1, M_2/M_1, and M/M_2 are objects of $\mathfrak{M}$.

We define a new category $Q(\mathfrak{M})$ having the same objects as $\mathfrak{M}$ in the following way. A morphism in $Q(\mathfrak{M})$ from M' to M is an isomorphism of M' with an $\mathfrak{M}$-subquotient of M. Such a morphism is the same as an isomorphism class of diagrams

(*) 

where i is a monomorphism with cokernel in $\mathfrak{M}$, and p is an epimorphism with kernel in $\mathfrak{M}$. The morphism in $Q(\mathfrak{M})$ are composed in the evident way. Thus given a morphism from M" to M' represented by the arrows i', p' in the diagram



its composition with (*) is represented by the arrows $i.pr_2$ and $p'.pr_1$.

Let $|Q(\mathcal{M})|$ denote the geometric realization of the nerve of the category $Q(\mathcal{M})$, the nerve being the semi-simplicial set whose n-simplices are chains of composable arrows of length n. The zero object 0 may be interpreted as a basepoint of this space, hence we can make the following

Definition. $K_i(\mathcal{M}) = \pi_{i+1}(|Q(\mathcal{M})|, 0)$ for $i \geq 0$.

In order to make this reasonable, note that for any M in $Q(\mathcal{M})$ there are two arrows

$$0 \rightrightarrows M$$

which result from viewing 0 as a subobject and as a quotient of M. Thus each object determines a loop in the space $|Q(\mathcal{M})|$. Using the standard description of the fundamental group of a semi-simplicial set in terms of a maximal tree, it is not difficult to show that by means of this correspondence, the fundamental group of $|Q(\mathcal{M})|$ is isomorphic to the Grothendieck group of the category $\mathcal{M}$.

Observe that the category $Q(\mathcal{M})$ depends only on $\mathcal{M}$ and the exact sequences of objects of $\mathcal{M}$, hence the preceding definition makes sense for any small category with a suitable notion of exact sequence. Also it is only necessary that $\mathcal{M}$ be equivalent to a small category, in order that $|Q(\mathcal{M})|$ be a well-defined homotopy type. For example, we can take $\mathcal{M}$ to be the category $\mathcal{P}_A$ of finitely generated projective left modules over the ring A, with the usual notion of exact sequence for modules. In this case we have the following basic result.

Theorem 1. *The loop space of* $|Q(\mathcal{P}_A)|$ *is homotopy equivalent to* $K_0A \times BGL(A)^+$, *where* K_0A *is the Grothendieck group of* $\mathcal{P}_A$. *Consequently* $K_i(\mathcal{P}_A) = K_iA$ for $i \geq 0$.

This theorem, and the three that immediately follow, are proved by a detailed cohomological study of categories of the form $Q(\mathcal{M})$.

Theorem 2. *Let* $\mathcal{P}$ *be a full subcategory of* $\mathcal{M}$ *such that*

(i) *For any exact sequence in* $\mathcal{M}$

(**) $0 \to M' \to M \to M'' \to 0$

we have

(a) $M', M'' \in \mathcal{P} \Rightarrow M \in \mathcal{P}$

(b) $M \in \mathcal{P} \Rightarrow M' \in \mathcal{P}$.

(ii) For every object M'' of $\mathcal{M}$, there exists an exact sequence (**) in $\mathcal{M}$ with M and M' in $\mathcal{P}$.

Then the induced map $|Q(\mathcal{P})| \to |Q(\mathcal{M})|$ is a homotopy equivalence.

Theorem 3. Let $\mathcal{A}$ be a small abelian category and let $\mathcal{B}$ be a full subcategory which is abelian and such that the inclusion functor is exact. Suppose also that every object of $\mathcal{A}$ admits a finite filtration whose quotients are objects of $\mathcal{B}$. Then the map $|Q(\mathcal{B})| \to |Q(\mathcal{A})|$ is a homotopy equivalence.

Theorem 4. Let $\mathcal{A}$ be a small abelian category, let $\mathcal{B}$ be a Serre subcategory, and let $\mathcal{A}/\mathcal{B}$ be the quotient category. Then $|Q(\mathcal{B})|$ is homotopy equivalent to the homotopy-theoretic fibre of the map $|Q(\mathcal{A})| \to |Q(\mathcal{A}/\mathcal{B})|$. Consequently, there is a long exact sequence

$$\to K_i(\mathcal{B}) \to K_i(\mathcal{A}) \to K_i(\mathcal{A}/\mathcal{B}) \xrightarrow{\partial} K_{i-1}(\mathcal{B}) \to .$$

§3. **Some applications.** If A is a left noetherian ring, let $\mathrm{Modf}(A)$ denote the abelian category of finitely generated left A-modules, and set

$$G_i A = K_i(\mathrm{Modf}(A)).$$

Recall that A is called left regular if it is left noetherian and if every object of $\mathrm{Modf}(A)$ is of finite projective dimension.

Theorem 5. If A is left regular, then $K_* A \cong G_* A$.

In effect, let $\mathcal{M}_i$ be the full subcategory of $\mathrm{Modf}(A)$ consisting of modules of projective dimension $\leq i$. Theorem 2 implies that $|Q(\mathcal{M}_{i-1})|$ is homotopy equivalent to $|Q(\mathcal{M}_i)|$ for each i, hence by a limit argument it follows that $|Q(\mathcal{P}_A)| \to |Q(\mathrm{Modf}(A))|$ is a homotopy equivalence, whence the theorem.

Theorem 6. _If_ I _is a nilpotent ideal in a left noetherian ring_ A, _then_ $G_*(A/I) \cong G_*A$.

Theorem 7. _Let_ $\mathfrak{A}$ _be a small abelian category in which every object has finite length._ _Then_

$$K_*(\mathfrak{A}) \cong \bigoplus_{j \in J} K_* D_j$$

where $\{X_j,\ j \in J\}$ _is a set of representatives for the isomorphism classes of simple objects in_ $\mathfrak{A}$, _and_ D_j _is the sfield_ $\mathrm{End}(X_j)$.

These result by applying Theorem 3 to the inclusion $\mathrm{Modf}(A/I) \to \mathrm{Modf}(A)$, and to the inclusion of the semi-simple objects in $\mathfrak{A}$.

Theorem 8. _If_ A _is a Dedekind domain with fraction field_ F, _then there is a long exact sequence_

$$\to K_i A \to K_i F \xrightarrow{\partial} \bigoplus_m K_{i-1}(A/m) \to K_{i-1} A \to$$

where m _runs over the set of maximal ideals of_ A.

This follows from Theorem 4 with $\mathfrak{B}$ the full subcategory of torsion modules in $\mathfrak{A} = \mathrm{Modf}(A)$, together with Theorems 5 and 7.

The transfer: If A is any ring, let $\mathfrak{M}_i$ be the full subcategory of the category of left A-modules consisting of those modules which admit resolutions of length $\leq i$ by objects of $\mathcal{P}_A$. Applying Theorem 2 inductively, one sees that $|Q(\mathcal{P}_A)| \to |Q(\mathfrak{M}_i)|$ is a homotopy equivalence for all i. Thus if $f: A \to B$ is a ring homomorphism such that B is an object of $\mathfrak{M}_i$ for some i, then restriction of scalars provides a functor $Q(\mathcal{P}_B) \to Q(\mathfrak{M}_i)$, and hence gives rise to a homomorphism

$$f_* : K_i B \to K_i A.$$

§4. Graded rings, filtered rings, and the homotopy axiom.

Theorem 9. _Let_ $A = \bigoplus_{n \geq 0} A_n$ _be a graded ring such that_

(i) A _is left noetherian_

(ii) A _is flat as a right_ A_0_-module_

(iii) A_0 is of finite Tor-dimension as a right A-module.
Let Modfgr(A) be the category of finitely generated graded left A-modules $M = \bigoplus_{n\geq 0} M_n$. Then $K_i(\text{Modfgr}(A)) \cong G_iA_0 \otimes_{\mathbf{Z}} Z[T]$ for all i.

Theorem 10. Let $A = \bigcup_{n\geq 0} F_nA$ be a ring with an increasing filtration such that $1 \in F_0A$ and $F_iA.F_jA \subset F_{i+j}A$. Suppose that the associated graded ring $gr(A) = \oplus F_nA/F_{n-1}A$ satisfies the hypotheses of Theorem 9. Then $G_*(F_0A) \cong G_*A$.

Sketch of proof: Let t denote the element of degree one of the graded ring $A' = \oplus F_nA$ represented by $1 \in F_1A$, and let $\mathcal{B}$ be the Serre subcategory of $\mathcal{A} = \text{Modfgr}(A')$ consisting of modules on which t is nilpotent. Then $\mathcal{A}/\mathcal{B}$ is equivalent to Modf(A). By Theorem 3, $|Q(\mathcal{B})|$ is homotopy equivalent to $|Q(\mathcal{B}')|$, where $\mathcal{B}'$ is the subcategory of $\mathcal{B}$ consisting of $A'/tA' = gr(A)$ modules, hence the exact sequence of Theorem 4 takes the form

$$\rightarrow K_i(\text{Modfgr}(gr\ A)) \xrightarrow{u} K_i(\text{Modfgr}(A')) \rightarrow K_i(\text{Modf}(A)) \rightarrow .$$

By the preceding theorem, the source and target of u are isomorphic to $G_i(F_0A) \otimes_{\mathbf{Z}} \mathbf{Z}[T]$; one shows that u is multiplication by $T - 1$, whence the result.

As a corollary one has the first part of the following.

Theorem 11. If A is left noetherian, then

(a) $G_iA \cong G_i(A[T])$

(b) $G_i(A[T, T^{-1}]) \cong G_iA \oplus G_{i-1}A$.

When A is left regular, G_* may be replaced by K_* in this theorem. According to Gersten [3], the isomorphism $K_*A \cong K_*(A[T])$ for left regular rings signifies that the Karoubi-Villamayor K-groups coincide with the ones considered here for such rings. Here is another application of Theorem 10.

Corollary. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field k and $U(\mathfrak{g})$ its enveloping algebra. Then $K_*(k) \cong K_*(U(\mathfrak{g}))$.

§5. Higher K-theory for schemes. If X is a noetherian scheme, let $G_*(X)$ be the K-groups of the abelian category of coherent sheaves on X, defined as in §2. Then, at least if we restrict to schemes having ample invertible sheaves, the preceding arguments permit one to define maps $f_* : G_*(X) \to G_*(Y)$ for a proper map $f : X \to Y$, (resp. $f^* : G_*(Y) \to G_*(X)$ when f is of finite Tor-dimension) with the usual properties. In addition, one has a long exact sequence

$$\to G_i(X - U) \to G_i(X) \to G_i(U) \to G_{i-1}(X - U) \to$$

when U is an open subscheme of X, the homotopy axiom:

$$G_i(X) \cong G_i(X \times_{\mathrm{Spec}\, \mathbf{Z}} \mathrm{Spec}\, \mathbf{Z}[T])$$

and the projective bundle theorem:

$$G_i(\mathbf{P}E) \cong G_i(X) \otimes_{K_0(X)} K_0(\mathbf{P}E)$$

where $\mathbf{P}E$ is the projective fibre bundle associated to a vector bundle E over X, and where K_0 is the Grothendieck group of vector bundles. Finally, by filtering the category of coherent sheaves on X according to the dimension of the support, one obtains a spectral sequence

$$E^1_{pq} = \bigoplus_{\dim(x) = p} K_{p+q} k(x) \Rightarrow G_{p+q}(X)$$

relating $G_*(X)$ to the K-groups of the various residue fields of the points of X, which generalizes Theorem 8.

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Massachusetts Institute of Technology
Cambridge, Massachusetts 02139
U. S. A.