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Algebraic Topology and Elliptic Operators*

M. F. ATIYAH
University of Oxford

Introduction

This is a purely expository article in which I shall attempt to explain, with a minimum of technicalities, the deep underlying connections between the *analysis of elliptic operators* and the *topology of the general linear groups*.

Let me begin with a few brief historical remarks. The theory of elliptic equations begins with the classical Laplace and Cauchy-Riemann equations, and it can be generalized in two different directions:

(A) we can consider more general elliptic systems,

(B) we can consider the classical operators but on more general manifolds.

Under (B) we may include all the topological-transcendental study of algebraic varieties initiated by Hodge and extensively developed by Kodaira-Spencer, Cartan-Serre, Hirzebruch and others. In this program one of the major tasks was the investigation of global topological invariants of manifolds. Because a manifold is "infinitesimally linear" these investigations are essentially concerned with the linear groups $GL(n, \mathbf{R})$ and $GL(n, \mathbf{C})$. In fact a large part of the work in algebraic and differential topology of the past 20 years has been concerned basically with the topology of the linear groups.

Most of the earlier work under (A) was concerned with the qualitative side, extending basic analytical results to general operators. Recently, however, there has been a certain fusion between (A) and (B) arising from the attempt to obtain for general elliptic systems some of the quantitative results available for the classical systems. The reason why this attempt has been successful lies in the fact that the topological properties of the linear groups which were so extensively developed in (B) proved to be precisely the right tools for (A). This is what I hope to explain.

1. Topology of the Linear Groups

Let me now try to explain some of the basic facts about the topology of the linear groups $GL(n, \mathbf{C})$—I concentrate mainly on the complex case but shall make

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a few remarks at the end about the real case. I shall begin with what is surely the one piece of algebraic topology which every mathematician knows.

Let $f: S^1 \to C^*$ be a continuous map from the circle S^1 into the non-zero complex numbers C^*. In other words, we have a closed path in the plane, missing the origin. The following facts are then well-known:

(i) f has a "winding number" or degree (the number of times the path "goes round" the origin);

(ii) this degree (written $\deg f$) is invariant under continuous deformation;

(iii) $\deg f$ is the only such invariant, i.e., f can be deformed into g if and only if $\deg f = \deg g$;

(iv) there exists a map f with any given degree.

There are many ways of defining or computing $\deg f$, depending on the context and the techniques one wants to use. Briefly these may be summarized as follows.

Geometric. We replace f by $g = f/|f|$ which maps $S^1 \to S^1$, then we approximate g by a differentiable map and count (algebraically) the number of points in the inverse image of a general point.

Combinatorial. We approximate by a piece-wise linear path and then use combinatorial methods.

Differential. We approximate by a differentiable f and then put $\deg f = (1/2\pi i)\int df/f$.

Algebraic. We approximate by a finite Fourier series $f = \sum_{n=-k}^{k} a_n z^n$ and then put

$$\deg f = N(f) - P(f),$$

where N, P are the number of zeros and poles in $|z| < 1$.

One may now ask what happens in higher dimensions. Of course many generalizations of the problem are possible depending on one's point of view, but there is one very beautiful generalization due to Bott [10] which I proceed to explain.

We consider continuous maps

$$f: S^{n-1} \to GL(N, \mathbf{C}), \qquad 2N \geqq n,$$

where S^{n-1} is the unit sphere in $\mathbf{R}^n$. The case discussed above corresponds to $n = 2$, $N = 1$. The theorem of Bott is then as follows:

Theorem. *If n is odd, every map f can be deformed to a constant map. If n is even, we can define an integer, called* $\deg f$, *and f can be deformed into another map g if and only if* $\deg f = \deg g$; *moreover there exists a map with any given degree.*

Again, as in the case $N = 1$, several definitions of deg f are possible. First we have a *geometrical* definition. This is simplest to explain when $2N = n$. In this case the first column of the matrix f defines a map

$$f_1 = S^{n-1} \to \mathbf{C}^N - \{0\}$$

so that $g = f_1/|f_1|$ is a map $S^{n-1} \to S^{n-1}$. This map has a degree—the number of points in $h^{-1}(P)$, where h is a differentiable approximation to g and P is a general point. We then define

$$\deg f = \frac{(-1)^{N-1} \deg g}{(N-1)!} .$$

The reason for the unexpected factor $(N - 1)!$ is that deg g turns out always to be divisible by $(N - 1)!$. When suitably normalized in this way deg f takes on all integer values. The sign $(-1)^{N-1}$ is put in for minor technical convenience. When $2N > n$ one can show that f can always be deformed into a map g so that

$$g(x) = \begin{pmatrix} h(x) & 0 \\ 0 & 1 \end{pmatrix},$$

where $h(x) \in GL(\frac{1}{2}n, \mathbf{C})$. We then define $\deg f = \deg h$ and it turns out to be independent of the choice of g.

There is also a *differential* definition of deg f. We put

$$\deg f = \int_{S^{n-1}} f^*\omega ,$$

where ω is a certain explicitly defined invariant differential form on $GL(N, \mathbf{C})$ and $f^*\omega$ is the induced form on S^{n-1}.

The algebraic definition by counting zeros and poles which I mentioned for the case $N = 1$ does not generalize in any obvious sense. On the other hand, in a very deep sense which I will explain later, it does have a generalization and one moreover that goes to the heart of the problem.

Let us pause here to contemplate the situation. I think it is true to say that this theorem of Bott must rank as one of the real achievements of topology and it is certainly something of which everybody should be aware. The existence of the degree is of course fairly easy—this is *homology*, but the fact that maps of equal degree can be deformed into one another is highly nontrivial—this is *homotopy*. It is definitively non-intuitive, and we need only consider some other cases of homotopy to see how fortunate we are. Thus the homotopy classes of maps of spheres into spheres are extraordinarily complicated and still, in general, unknown. The same is true of maps

$$S^{n-1} \to GL(N, \mathbf{C})$$

when $2N < n$.

To understand Bott's theorem a little better it is helpful to examine the relation between different dimensions. Suppose then that

$$f : S^{n-1} \to GL(N, \mathbf{C}) , \qquad g : S^{m-1} \to GL(M, \mathbf{C})$$

are two given maps. We agree to extend f so that it is defined on the whole of $\mathbf{R}^n$ by

$$f(\lambda x) = \lambda f(x) , \qquad x \in S^{n-1}, \lambda \geqq 0 ,$$

and similarly for g. Then f, g are now continuous matrix-valued functions defined on $\mathbf{R}^n$, $\mathbf{R}^m$, respectively. We then define a matrix-valued function h on $\mathbf{R}^{n+m}$ by

$$h(x,y) = \begin{pmatrix} f(x) \otimes 1_N , & -1_M \otimes g^*(y) \\ 1_M \otimes g(y), & f^*(x) \otimes 1_N \end{pmatrix} ,$$

where 1_M denotes the identity transformation of $\mathbf{C}^M$ and $f^*(x)$ is the transposed conjugate of the matrix $f(x)$. Thus $h(x,y)$ is a $2MN \times 2MN$ matrix. It is easy to check that for $(x, y) \neq (0, 0)$ it is nonsingular and so it defines a continuous map

$$S^{m+n-1} \to GL(2MN, \mathbf{C})$$

which we may denote by $f * g$ to indicate that it is a kind[1] of "product" of f and g. If m and n are both even, we have the simple multiplicative formula

$$\deg (f * g) = (\deg f)(\deg g) .$$

Thus if $a : S^1 \to GL(1, \mathbf{C})$ is the standard map of degree 1 given by

$$a(z) = z , \qquad z \in \mathbf{C} , |z| = 1 ,$$

then the map

$$a_n : S^{2n-1} \to GL(2^{n-1}, \mathbf{C})$$

defined by $a_n = a * a * \cdots * a$ (n times) has degree 1. It is thus the *generating map* in this dimension, i.e., it defines a generator of the (infinite cyclic) group of homotopy classes of maps $S^{2n-1} \to GL(2^{n-1}, \mathbf{C})$.

There are now many different proofs of Bott's theorem. Later I shall comment on one of these proofs but for the present let me just say that *all* the proofs proceed by induction on n, or rather by an inductive step from n to $n + 2$. What one has to prove in this step is that $f \to (f * a)$ gives an isomorphism from the homotopy group in dimension $2n - 1$ to that in dimension $2n + 1$.

[1] The reader who is puzzled by this peculiar looking product can consult [5] where it arises in a natural manner.

Remark. If $N < n$, the degree of a map $S^{2n-1} \to GL(N, \mathbf{C})$ should be defined as zero, because the degree of the composite map

$$S^{2n-1} \to GL(N, \mathbf{C}) \to GL(n, \mathbf{C})$$

is easily seen to be zero.

2. Elliptic Operators

In the theory of elliptic differential operators one is led naturally to a larger class of integro-differential operators (including the inverse or Green's operator). These are now called psuedo-differential operators and are most conveniently studied via Fourier transforms. They have local integral representations of the following type:

$$P\phi(x) = (2\pi)^{-n} \int e^{i\langle x, \xi\rangle} p(x, \xi) \hat{\phi}(\xi)\, d\xi \,.$$

Here $p(x, \xi)$ is a smooth function having suitable asymptotic properties[2] as $\xi \to \infty$ (uniformly for bounded x) and $\hat{\phi}$ is the Fourier transform of ϕ. For a differential operator p is just a polynomial in ξ whose coefficients are smooth functions of x. If ϕ is vector-valued, then p is matrix-valued.

Let p_r denote the highest order terms of the asymptotic expansion of p. Then P is said to be elliptic of order r if $p_r(x, \xi) \in GL(N, \mathbf{C})$ for all x and all $\xi \neq 0$. Thus for fixed x we have a continuous map $S^{n-1} \to GL(N, \mathbf{C})$ given by $\xi \to p_r(x, \xi)$. Hence if n is even this map has a *degree*. This is independent of x (if our manifold is connected) and may be called the *local degree of* P. For example the local degree of the Cauchy-Riemann operator $d/d\bar{z}$ is clearly equal to 1, while the Laplace operator has local degree 0.

To get an interesting global problem we have usually to impose suitable boundary conditions for P. If we are on a compact manifold without boundary, then of course there is no question of boundary conditions. The same is true for operators of order zero on $\mathbf{R}^N$ which are "equal to the identity at infinity" in the following sense:

(1) There exists a compact set $K \subset \mathbf{R}^N$ so that $\phi P \psi = \phi\psi$, whenever ϕ or ψ have support in $\mathbf{R}^N - K$.

If $P = p(x, D)$, then condition (1) implies

$$p(x, \xi) = 1 \qquad \text{for} \qquad x \notin K \,. \tag{2}$$

In fact, (1) is equivalent to (2) and $(2)^t$, where $(2)^t$ denotes the corresponding condition for the transpose (or adjoint) operator P^t of P. Because Euclidean space

[2] See [12] or [13] for precise definitions.

is simpler than a general manifold I shall, for the moment, restrict myself to this case.

Suppose then that P is elliptic of order zero and satisfies (1). Then, if k is any constant so that

$$|x| \geqq k \Rightarrow x \notin K ,$$

the leading term $p_0(x, \xi)$ of $p(x, \xi)$ will be a non-singular matrix for all (x, ξ) with $|x| + |\xi| \geqq k$. In fact, for $\xi \neq 0$ this is a consequence of ellipticity and for $\xi = 0$ we have $|x| \geqq k$, so that $x \notin K$ and therefore, by (2), $p_0(x, \xi) = 1$. Thus p_0 defines a continuous map

$$p_0 : S^{2n-1} \to GL(N, \mathbf{C}) ,$$

where S^{2n-1} is the sphere of radius in $\mathbf{R}^{2n}$ (the x, ξ-space). This degree of p_0 is independent of k and may be called the *global degree* of the operator P. It is quite different from the local degree defined earlier. In the first place, the global degree is defined for all values of n, whereas the local degree is only defined for *even* values. In the second place, if P is an operator of the kind we are now considering and n is even, then the local degree is necessarily zero (because it is independent of x and, for $|x| > k$, $p_0 = 1$). In fact we can say that the global degree is only defined because the local degree is zero.

These simple facts about local and global degrees for Euclidean space turn out to be typical of what happens in the case of quite general boundary conditions of Shapiro-Lopatinskii (or "coercive") type[3] on general manifolds. In fact, for an elliptic operator P to admit any boundary conditions of this type, even locally, the local degree of P must be zero. In other words, there are topological conditions which must be satisfied for P to admit Shapiro-Lopatinskii boundary conditions. This was known and is rather obvious for $n = 2$: for example $d/d\bar{z}$ has local degree 1 and does not admit such boundary conditions. In the literature on the subject, $n = 2$ seemed to be regarded as a rather special case. In fact the contrary is true: it is typical.

If P does admit a Shapiro-Lopatinskii boundary condition B (the local degree being therefore zero), then the pair (P, B) together define a global topological invariant which generalizes the global degree.

An interesting point in this connection is that, in attempting to understand the topological significance of boundary conditions, Bott and I were inevitably led to a new and elementary proof of Bott's theorem [4]. This new proof is I believe the key to a real understanding of the relation between elliptic operators and the linear groups. Moreover, its significance for pure algebraic topology is quite dramatic. It opens up the possibility of developing large parts of algebraic topology on the basis of linear algebra rather than on the orthodox theory of simplicial complexes and homology.[4]

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[3] See [3] for precise definitions and for proofs of the statements which follow.

[4] This program is followed in [1].

But we digress. Let us return to the special case of elliptic operators P in Euclidean space satisfying the condition (1). One of the basic results of the classical theory of elliptic operators which remains valid here is that the space of solutions of the equations $P\phi = 0$ is finite-dimensional. The adjoint P^* of P being of the same type as P, it follows that $P^*\psi = 0$ has a finite-dimensional space of solutions also. The difference between these two dimensions is called the *index* of P. Thus

$$\text{index } P = \dim \operatorname{Ker} P - \dim \operatorname{Ker} P^* .$$

The special interest of the index is that it is *unchanged by continuous deformation* of P. It is therefore reasonable to ask what relation it has to our topological invariant, the global degree.

Let us denote by $\text{Ell}(\mathbf{R}^n, \mathbf{C}^N)$ the space of all elliptic operators P acting on the space of functions $\mathbf{R}^n \to \mathbf{C}^N$ and satisfying[5] condition (1). Then the process of assigning to each such P the function $p_0(x, \xi)$ defines a continuous map

$$\sigma : \text{Ell}(\mathbf{R}^n, \mathbf{C}^N) \to \text{Map}(S^{2n-1}, GL(N, \mathbf{C})) ,$$

where $\text{Map}(A, B)$ denotes the function space of all continuous mappings $A \to B$. It is elementary to show that σ induces a one-one correspondence between the *connected components* of these two spaces. In other words, the homotopy classes of operators correspond precisely to elements of the $(2n - 1)$-th homotopy group of $GL(N, \mathbf{C})$. Since

$$\text{index } P = \text{index } (P \oplus I) ,$$

where I is the identity operator, we can always increase N without altering the index (or the global degree). Thus there is no loss of generality in assuming $N \geqq n$. We are then in a position to apply Bott's theorem asserting that $\deg f$ is essentially the only homotopy invariant of a map

$$f : S^{2n-1} \to GL(N, \mathbf{C}) .$$

This implies that index P must be some function of degree $\sigma(P)$, the global degree of P. Since "index" and "degree" are both rather trivially additive for direct sums, it follows that

$$\text{index } P = C_n \text{ degree } \sigma(P)$$

for some integer C_n independent of P. To compute this constant C_n it is sufficient to compute index P for an operator P whose symbol $\sigma(P)$ is equal to the Bott generator a_n. This may be done as follows. We first establish a multiplicative property of the index[6]: if $\sigma(R) = \sigma(P) * \sigma(Q)$, then index R = index $P \cdot$ index Q.

[5] We consider a fixed compact set K in (1) and by a change of scale we can assume K contained in the unit ball.

[6] See [14] for the case of a closed manifold. Only minor modifications are needed for the case of Euclidean space.

Here $*$ is the operation defined in Section 1. Thus if $P \in \text{Ell}(\mathbf{R}^n, \mathbf{C}^N)$, $Q \in \text{Ell}(\mathbf{R}^n, \mathbf{C}^M)$, then $R \in \text{Ell}(\mathbf{R}^n, \mathbf{C}^{2NM})$. Since degree $\sigma(P)$ is also multiplicative in the same sense, and since

$$a_n = a * a * \cdots * a \ (n \text{ times}) ,$$

it follows that $C_n = (C_1)^n$. It remains therefore to calculate C_1 and for this one has only to evaluate the index of an operator P on $\mathbf{R}^1$ with $\sigma(P) = a_1$. This is in fact a classical example and one finds[7] $C_1 = 1$. Thus we have

THEOREM. *For any elliptic pseudo-differential operator on $\mathbf{R}^n$, satisfying condition* (1), *the index is equal to the global degree.*

Using one or the other of the explicit definitions of degree, this theorem gives us an explicit formula for the index. Alternatively (in view of the theorem) *we could regard the index as providing an analytical definition of the degree*! This is not quite so bizarre as it sounds. In the first place it is the only definition for which the degree is *a priori* an integer: in the geometric definition we divided by $(n - 1)!$ and in the differential definition the formula is given by an integral (hence is *a priori* a real number). In the second place, this analytical definition is in some sense the appropriate generalization of the algebraic definition of degree (which exists only for $n = 2$). The superficial formal analogies are obvious, both degrees being a difference of two positive integers. In fact the analogy goes much deeper, because the number of zeros or poles of a meromorphic function in $|z| < 1$ can quite naturally be interpreted as the dimension of the space of solutions of a suitable differential equation. Finally, for certain natural generalizations which we shall discuss in Section 3, this analytical definition extends in a way which the other definitions do not.

On the debit side one must of course admit that the index of an operator is usually a less computable quantity than say an integral. But for many theoretical purposes actual computations are not relevant and the analytical definition has many theoretical advantages.

This is perhaps a convenient occasion to comment on the various places in the literature where a proof of the above theorem is given. The first proof for general n is contained in[8] [7], the details of which are elaborated in [14]. Because however these papers were concerned with the case of general manifolds more sophisticated notation and machinery were employed than is necessary for the case of Euclidean space (or a sphere). If specialized down to the case of a sphere the proofs of [7], [14] coincide essentially with the one I have outlined here. Only two points of

[7] In fact we have $C_1 = \pm 1$ depending on sign conventions which we shall ignore here.

[8] There is also a paper by Vol'pert on the subject (Doklady Akad. Nauk SSSR, Vol. 152, 1963, pp. 1292–1293 or Soviet Math. Doklady, Vol. 4, 1963, pp. 1540–1542). Unfortunately the paper is erroneous. In the first place Bott's theorem is assumed, as if it were obvious (no statement or reference being given). Moreover, the formula given is incorrect, the factor $(n - 1)!$ being omitted. Both these are understandable errors if one believes that the general case is identical with $n = 2$.

difference require special comment. First the more difficult part of [7], concerned with cobordism invariance, is not needed for spheres. Secondly there is the minor difference between Euclidean space and the sphere. The idea of treating operators in Euclidean space which are "equal to 1 at infinity" is due to Seeley [15].

Because the proof in [7] appeared to be complicated, various authors, [15], [11], [9], have attempted to give simpler or more elementary proofs for the case of Euclidean space. These different proofs are different only in their use and presentation of algebraic topology—the analysis is basically common ground. The topology which these authors use is of the more classical variety—homology, Hurewicz theorem, Serre's theorems on the homotopy of spheres, etc.—combined sometimes with Bott's theorem and its homological corollaries. In my view, although these parts of topology are older, they are far from being more elementary than Bott's theorem. The reader may compare the length of [4], which is a self-contained account of all that I have really used here, with standard advanced treatises on algebraic topology. More important still, I firmly believe that Bott's theorem is not only more elementary but also that it is much more *relevant* to the index problem. This has been the whole emphasis of my presentation.

3. Wider Implications

I have concentrated so far on operators in Euclidean space because this is easier to explain and because I think it goes very much to the heart of the matter. The justification for saying this must of course come from showing that much more general problems get solved in a similar way. This is in fact true and I would like to discuss a number of such problems.

First the *general index problem* for arbitrary elliptic operators on closed[9] manifolds has been solved. In addition to the proof in [7], [14] there is now a second proof, which will appear in [8], and which is superior in many important respects. The basic idea is quite easily explained. Given an elliptic operator P on a closed manifold X, we imbed X in Euclidean space E and construct an operator Q on E "equal to 1 at infinity" and such that index $P =$ index Q. The problem is thus reduced to the case studied in Section 2. The construction of Q may very roughly be described as follows. We take an operator A, defined in a tubular neighbourhood N of X in E, which gives the generating symbol in each normal plane N_x, $x \in X$. We then form $Q = P * A$, where $*$ denotes an operation like that described in Section 1. A refinement of the multiplicative property of the index gives

$$\text{index } Q = \text{index } P \cdot \text{index } A_x = \text{index } P\,.$$

The index problem for manifolds with boundary can be reduced to that on closed manifolds along the lines indicated in [3].

[9] i.e., compact without boundary.

The formulae for the index obtained in this way can, like the global degree of Section 2, be written as integrals or alternatively they can be given in homological form. It is an interesting fact that some of the most important invariants of differentiable manifolds discovered by topologists turn out to be indices of elliptic operators. This helps to explain some of their properties, but much remains to be understood in this direction.

The solution of the general index problem just described may be regarded as an application of algebraic topology to a problem in analysis. Conversely one can use the analysis to help the topology as the following example shows.

Let G be a compact Lie group (for example a finite group) and let V, W be two (finite-dimensional) complex representation spaces of G. Then the unit sphere $S(V)$, in an invariant metric, and $GL(W)$ are both G-spaces in a natural way: the action on $GL(W)$ is conjugation, i.e., $T \to gTg^{-1}$. What can we say about continuous G-maps

$$f: S(V) \to GL(W),$$

i.e., maps satisfying the condition

$$f(gx) = gf(x)g^{-1}, \qquad x \in V, g \in G? \tag{3}$$

What are the G-homotopy[10] classes of such maps? This problem is a natural generalization of the problem solved by Bott's theorem, at least if W is "large". Let us therefore define the "stable group":

$$A(V) = \lim_{\overrightarrow{W}} [S(V), GL(W)],$$

where W runs over all representation spaces, directed by inclusion, and [,] denotes G-homotopy classes of mappings. If $G = 1$, then Bott's theorem asserts that $A(V) \cong \mathbf{Z}$, the group of integers. The global theorem is then[11]:

THEOREM. *An element of $A(V)$ has one degree, written* $\deg_\chi$, *for each irreducible character χ of G. Two elements ϕ, ψ are equal if and only if*

$$\deg_\chi \phi = \deg_\chi \psi \qquad \textit{for all} \qquad \chi.$$

Finally a family of integers n_χ occur as the degrees of some $\phi \in A(V)$ if and only if

(i) $n_\chi = 0$ *for all but a finite set of* χ,

(ii) $(\Sigma n_\chi \chi(g)) \det (1 - \rho(g)) = 0$ *for all $g \in G$, where $\rho(g) \in GL(V)$ defines the action of G on V.*

Remark. Note in particular that, if G has a fixed vector in V, $\det (1 - \rho(g)) = 0$ and so condition (ii) is vacuous. Thus the n_χ are arbitrary, as was the case when there was no group.

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[10] i.e., homotopies preserving condition (3).

[11] For the proof the reader may consult [6].

The proof of Bott's theorem cannot be generalized immediately because it proceeds by induction on dim V, and for a general non-commutative group G the representation V need not decompose into one-dimensional subspaces. However, using the index of elliptic operators one can get around this difficulty. I cannot go into details but let me just say that, of the various definitions of degree mentioned earlier, the only one which generalizes in a completely satisfactory manner is the analytical definition. For simplicity assume $V = U \otimes_{\mathbf{R}} \mathbf{C}$ is the complexification of a real representation space U, and let $\mathrm{Ell}_G(U, W)$ denote the space of elliptic operators as in Section 2 which are in addition G-invariant (in an obvious sense). Then the symbol $\sigma(P)$ of such an operator is a G-map

$$S(V) \to GL(W) .$$

The spaces Ker P and Ker P^* will be representation spaces of G and so we can define

$$\mathrm{index}_\chi P = d_\chi(\mathrm{Ker}\, P) - d_\chi(\mathrm{Ker}\, P^*) ,$$

where d_χ denotes the number of times the representation χ occurs. We can then define

$$\deg_\chi \sigma(P) = \mathrm{index}_\chi P .$$

I am firmly convinced that the relation between the analysis and the topology in all these questions is quite fundamental. One of my reasons for believing this is that time and time again we have found that the analysis has inexorably led to certain topological considerations which have turned out in the end to be just the right ones. I have already alluded to the boundary value problems as one instance. Let me conclude with another example which is more recent and very instructive.

So far I have used only complex numbers. We can however consider real operators (e.g., differential operators with real coefficients) and the real linear group $GL(N, \mathbf{R})$. It is natural to look for relations between these which refine the ones we have been investigating for the complex case. Now Bott [10] has also determined the homotopy groups π_{n-1} of maps

$$S^{n-1} \to GL(N, \mathbf{R}) , \qquad N \text{ large} .$$

These are periodic in n, with period 8, and take the values:

$$\begin{array}{rcccccccc} n = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8, \\ \pi_{n-1} = & Z_2 & Z_2 & 0 & \mathbf{Z} & 0 & 0 & 0 & \mathbf{Z}, \end{array}$$

where Z_2 is the group of order 2. This suggests looking for mod 2 analytic invariants. It is not difficult to find some. Let P be real elliptic and skew-adjoint. Then index $P = 0$ and so is not interesting. But dim (Ker P) mod 2 is invariant under continuous deformation, because the non-zero eigenvalues of P occur in complex conjugate pairs: then if an eigenvalue $\lambda \to 0$ its conjugate $\bar{\lambda} \to 0$ and dim Ker P jumps by 2.

It is now natural to try and relate this mod 2 invariant of P with the groups of order 2 in Bott's theorem. All initial attempts to do this proved unsuccessful. The reason emerged later when Singer pointed out to me that, if P is real, $p(x, \xi)$ is *not* real but satisfies instead

$$p(x, -\xi) = \overline{p(x, \xi)},$$

because it is defined by Fourier transforms. This suggests that we should interpret the symbol of a real operator as a map

$$f : S^{2n-1} \to GL(N, \mathbf{C})$$

with the condition $f(-\xi) = \overline{f(\xi)}$. This turns out to work magnificently and one obtains the desired tie-up between the analytical and topological mod 2 invariants. Moreover, as a by-product, I was led to a new and much simpler topological approach to the real Bott theorems, [2]. Thus the analysis in this case was of great help in suggesting the most fruitful topological viewpoint.

These mod 2 invariants reinforce my earlier remarks that the analytical definition of degree is superior to the geometric or differential definitions. In fact no geometric or differential definition of Bott's mod 2 invariant is known. Moreover it is known that this invariant is definitely not of a homological type (even with mod 2 coefficients). Since all known integral formulae in this context are essentially homological, it would appear unlikely that one could compute Bott's invariant by integral methods. However this remains an interesting open problem.

To conclude let me just say that the analysis and topology are now inextricably mixed and one should perhaps refer to this part of mathematics as "elliptic topology".

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