Algebraic Topology

Ross/Indiana 2025

July 12–13th

Connor and Aareyan

(Notes by Jenny, Timothy, Kaka, Akhil)

Algebraic Topology

1 Math Until We Die

- Stolen straight from Canada/USA Mathcamp.
- We do it better.
- Proof Indiana > Ohio.

2 Revolution

For 70 years, Ross has been a number theory program. Today, we stage a revolution. For the next 10 hours, this is a topology program. However, Connor is a fake revolutionary. He is a number theorist and not a topologist. He hates shapes. Here is a squiggle.

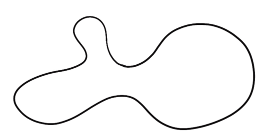


Figure 1: Squiggle

That's complicated. On the other hand, $2 + (-1) = 1$ is easy.

The integers have a finite list of axioms (9 of them), and those are all we need to work with them. But, the squiggle is a situation with a very *infinite* amount of information.

No one has any idea what is going on in the squiggle. There's an infinite number of points, and how the hell are we supposed to work with that?

There's a solution to this problem, and that's called coordinates—analytic geometry. Instead of just drawing a random blob, let's draw a square bounded by the coordinates $(0, 0)$ and $(2, 2)$. Going back to finite information is what makes geometry doable.

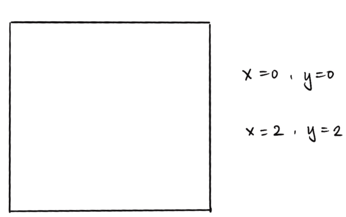


Figure 2: Square

It's quite difficult to calculate the area of a squiggle, but at least for squares, we have finite information to work with. For instance, as the side lengths are each 2, then $2 \times 2 = 4$. The same cannot be done for a squiggle, but that's not the point.

The idea is that analytic geometry is very good for calculating area and lengths, but we'll explore how there are other qualitative things that it is not as good for.

Say we have a circle (just the boundary) as opposed to the filled-in circle (a disk). Well, the first circle has a hole in it. But, how do we say that mathematically?

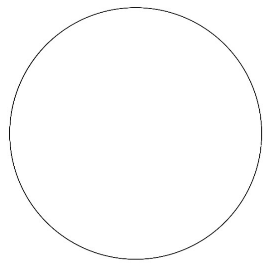


Figure 3: Circle

By the way, we write the circle as S^1, which is the sphere (circle) in one dimension.

Sure, we have intuition, but there's no simple way to say what that hole is. It's still a very infinite situation, and there's no number I can point to and say "that number is proof there is a hole."

Also, some holes seem to be different than others. Spheres, by default, are not filled in, so let's start with a sphere, which is just a big hole.

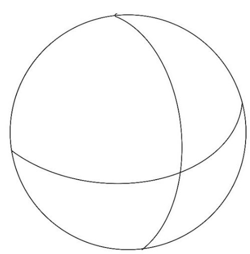


Figure 4: Sphere (hollow)

Is the circle hole different from the sphere hole? One hole is sort of like a lower dimension than the other. The latter is like volumetric in the sense that we could consider it a two-dimensional hole because the boundary is a two-dimensional object, while the former hole has a one-dimensional boundary. That'll factor in later.

Here's a cylinder. Like always, we don't want to consider the interior, or else the shape would just be the same as a ball. So, we get a sort of band.

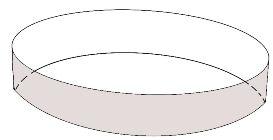


Figure 5: Cylinder band

Well, this is very similar to a circle, since we've just stretched it out.

The nature of the hole hasn't changed at all, but the dimension is the same as the shape. So, the nature of the hole seems to depend not on the dimension, but on something else.

Hopefully, we get the vibe. So, in honor of this anti-number theory revolution, here's **Set #19** with the peak of the number theory program.

Let's rip it in half (vertically).

And then, let's fold it into a cylinder and tape it up.

We now have a straw-like object. No drinking through it, Lucas.

Now, let's do something weird. Suppose we wrap it in a circle and tape it shut at one end. What do we get?

Aareyan

We get an elliptic curve!

If we inflate it a bit, we have a torus!

How can we visualize this? Well, let's start with a square and use arrows to represent the gluing.

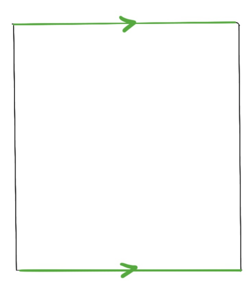


Figure 6: Starting square

First, we took the two arrows on top and bottom and glued them together. That gave us a cylinder.

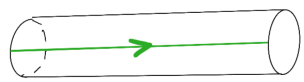


Figure 7: Taped cylinder

And then, we took the circular edges of our cylinder and stuck them together, giving us a torus.

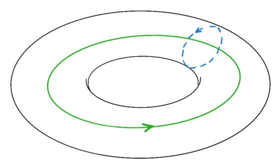


Figure 8: Torus

However, we'll actually draw that other arrow on the original square.

But, this isn't actually a torus. It's all crinkled up.

And also, we started with a square here, instead of a rectangle. The geometry is different. But, somehow in our soul (Dr. All likes to say heart of hearts), we feel this is more or less the same. We've gotten pretty much the same shape at the end.

And neither of these is a perfect mathematical torus. That isn't even a geometric object—we've just said two edges of the square are the same. That doesn't change the dimension or anything.

Well, we stole Dr. All's coffee mug (he no longer has property after the revolution). Inside, we sort of morph it smoothly by pulling up the interior and squishing the handle to get a torus.

How can we mathematically capture this notion?

2.1 Homeomorphisms

This is called a homeomorphism. It's sort of like a homomorphism, but different. Let's define it mathematically.

Homeomorphic

If $\underline{\overline{X}}$ and $\underline{\overline{Y}}$ are subsets of $\mathbb{R}^n$, we call them **homeomorphic** if there is a function $f : \underline{\overline{X}} \to \underline{\overline{Y}}$ if

- f is bijective (we should at least correspond their points),
- f is continuous (we don't make any cuts in our shape—if we have this) (we don't want our torus to be the same as our cylinder, so we are eliminating cuts)
- f^{-1} is continuous.

Well, we're not going to start from the real first principles. We'll assume notions like continuous and not define them rigorously.

Removing the continuity would allow us to cut open the torus and say that the torus is the same as a sphere. Now, the last thing is quite subtle. It might seem that the inverse is automatically continuous if the original function is, but that's not true. For instance, if we map a line segment (that is open at one endpoint but closed at the other, like $[0, 2\pi)$) to a circle S^1, then this function can definitely be made bijective. However, the inverse is not continuous where the endpoints match up.

Besides, we definitely want those two to be different—otherwise, that would violate so much intuition, like one having a hole and one not.

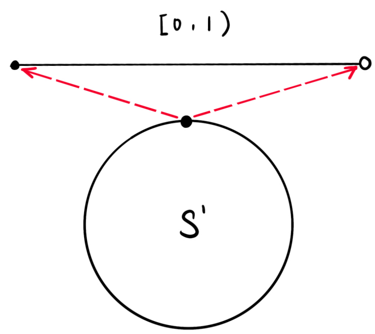


Figure 9: Two ends of the segment are mapped to adjacent points on the circle

This is super fundamental—it's our notion of what it means for two topological objects to be the same. In particular, we might think of a homomorphism, but this homeomorphism is much more like an isomorphism.

With that, let's go back to some more definitions.

Once again, we start off with paper constructions. Let's bring back the cylinder, as well as a Möbius band.

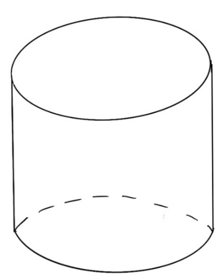


Figure 10: Another cylinder band

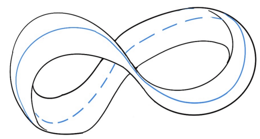


Figure 11: Möbius band

There's something sort of subtle about why these two are not homeomorphic. Can we even prove it? After all, they seem both to have holes in them.

For instance, we know that circles and lines are different since one has a hole and the other doesn't, but we can't even formalize that.

So, how do we show that a Möbius band and a torus are different? Let's draw a line segment and go around. In a sense, the Möbius band just has one side because the line segment starts and ends on "opposite sides" of the strip, while on a torus we stay on the same side. Just cut the paper and try it out.

So, a torus sort of has two sides, while a Möbius band only has one. Is there another way to describe this one-sidedness, which is a topological property that we can't quite describe? That's our challenge tonight—studying this rigorously.

Connor

This is the first part of my talk. Aareyan, where the hell are you?

Aareyan

I am here. Lock in. Sǔo Dìng (锁定)!

3 Holes

It'll take a while before we can formalize what holes are. Before we get to that, let's take the real number line ($\mathbb{R}^1$) and the Euclidean plane ($\mathbb{R}^2$). Can we find a homeomorphism between them?

I mean, we really hope that such is impossible, or else that would destroy all our intuition about dimension. I claim that a bijection is possible, but when we add in the condition that it must be continuous, it no longer is. The first part is unrelated to this course, but how do we rigorously show the second one?

Well, here's an angle of attack.

Let's delete a point on the line, which also deletes a point on the plane.

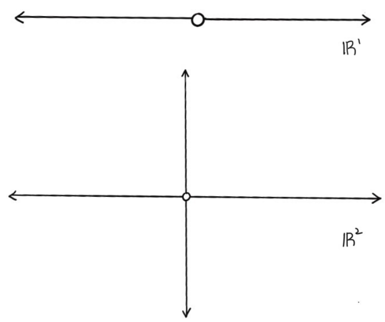


Figure 12: Punctured $\mathbb{R}^1$ and $\mathbb{R}^2$

On the one hand, the line is no longer connected (we can't get from one side to the other), while on the other hand the plane is (we can get around it).

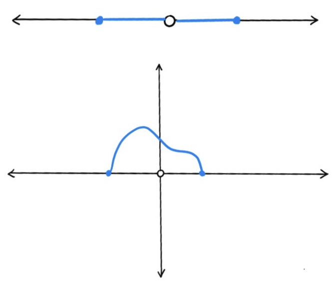


Figure 13: Paths in punctured $\mathbb{R}^1$ and $\mathbb{R}^2$

And, a homeomorphism should preserve that, right? So, let's formalize connectedness since that'll seem to be helpful. In our minds, connectedness means we can get from one point to another without going outside the shape.

Path

A **path** in $\underline{X}$ is a continuous map $\gamma\colon [0,1] \to \underline{X}$.

Once we know what a path means, connected just means there is a path between any two endpoints.

It turns out, though, that there are really scary spaces where our intuition breaks, and this definition no longer tells us what we want. In that case, we define different notions of connectedness—that's why we need to prefix this version. That being said, we are living in nice spaces where they are the same.

Path-Connected

So, **path-connected** means that there is some continuous path between any pair of points that goes through the two. Rigorously, $\forall x, \forall y, \exists \gamma\colon [0,1] \to \underline{X}, \gamma(0) = x \wedge \gamma(1) = y, \gamma$ is continuous.

Well, that allows us to show that $\mathbb{R}$ and $\mathbb{R}^2$ are the same, but that doesn't help for $\mathbb{R}^2$ and $\mathbb{R}^3$. Here's another idea.

Suppose we had a path that goes around the hole. Well, we can't just move it around nicely and shrink that path to a point (because of the hole in the middle), while we can easily do that on $\mathbb{R}^3$. Just shift it up a little!

So, what does it mean to deform a loop? We'll have a very similar description to the one we did for the subsets themselves.

Homotopy

Two paths $\gamma_0, \gamma_1 \colon [0,1] \to \underline{X}$ are said to be **homotopic** if there is some continuous map $H \colon [0,1] \times [0,1] \to \underline{X}$ so that $H(t,0) = \gamma_0(t)$ and $H(t,1) = \gamma_1(s)$.

By the way, we write $H(t,s)$ with two variables because we are mapping two things.

Basically, we are continuously deforming the path by letting the second parameter change.

Let's draw some paths to understand that. It's easier to understand topology when we draw some squiggles. Well, suppose we have two paths from a to b.

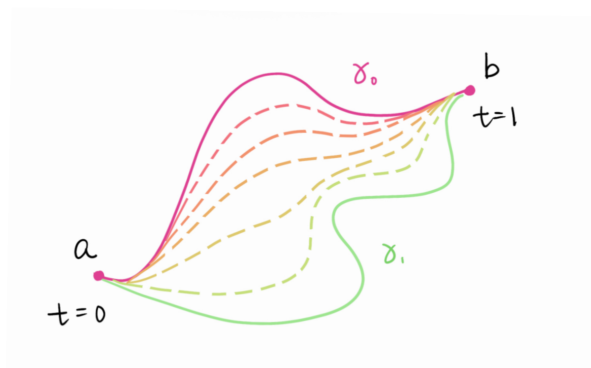


Figure 14: Visualization of a homotopy

The idea is that we can fill in paths in between (sort of like a sequence, but the paths are really close together) so that we can morph from one path to another. Each $H(t,s)$ for some fixed s is a new path that also varies for $t \in [0,1]$, and those represent the intermediate paths that we use to get from γ_0 to γ_1.

All that s is doing is choosing the in-between path (while t represents where on the path we are). The homotopy, by the way, isn't changing the endpoints of the paths, so $H(0,s) = a$ and $H(1,s) = b$ for any $s \in [0,1]$.

These don't relate to anything about shortest paths. Remember, topology is structured very loosely, and very crazy functions H could always exist. For instance, it could do something really weird and first go outside, but then go back inside.

Let's draw another example before we get on to rigorously defining holes and stuff.

Let's go to $\mathbb{R}^2$ for an idea. For instance, consider the trivial path $\gamma_0 \colon [0,1] \to \mathbb{R}^2$.

The trivial path is just going to be $t \mapsto 0$, so everything stays at the origin, always. By the way, since it starts and ends at the origin, let's actually call this the trivial loop.

And then, take some random arbitrary path $\gamma_1 \colon [0,1] \to \mathbb{R}^2$ that also starts and ends at the origin.

Aareyan

Pay attention to the things I write. I will "intentionally" miswrite things to make sure you guys are following. Everything I say is a podasip.

We claim that these two are homotopic, and it isn't too hard to write a rigorous proof. Let's define $H(t, s) := s\gamma_1(t)$, so basically all the intermediate paths are scalar multiples of γ_1. Well, this is definitely continuous, but we get back to the trivial loop when $s = 0$.

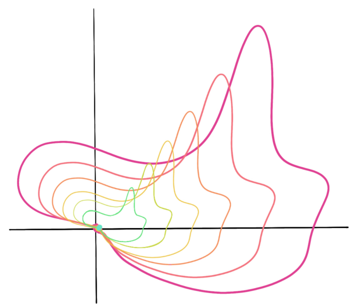


Figure 15: Every loop is homotopy equivalent to and can be shrunken down to the trivial loop

Note the important nuance in the definition of a homotopy. By convention, we will assume that for two paths to be homotopic, they must have the same endpoints (so $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$) but also that every intermediate function also shares those two endpoints. In fact, there are differences—some paths with the same endpoints are homotopic if we allow the intermediate endpoints to vary, but not otherwise (confirmed by Mustafa, who is scared by those). On the other hand, if we did consider paths with different endpoints, then every path would be homotopic to the trivial one using a similar procedure.

But anyway, every path is homotopic to the trivial path. And, since this path has the same start and end, we call it the trivial loop. And, you can use a similar argument to show that if we have any two points, any two paths between the two are homotopic.

Aareyan

I expect a full, rigorous write-up of this from you by tomorrow. What is your name again?

What

What?

Aareyan

What? That is an interesting name.

3.1 Fundamental Group

So now, we can talk about holes.

Fundamental Group

$\pi_1(\underline{\overline{X}}, x_0)$ is defined as $\{\gamma\colon [0,1] \to \underline{\overline{X}}$ up to homotopy, $\gamma(0) = \gamma(1) = x_0\}$. Basically, it's all the loops going around x_0 up to homotopy.

Hopefully, this fundamental group is going to capture this notion of holes. If there's a hole, then we won't be able to "pull" our loop through the hole—it'll get stuck.

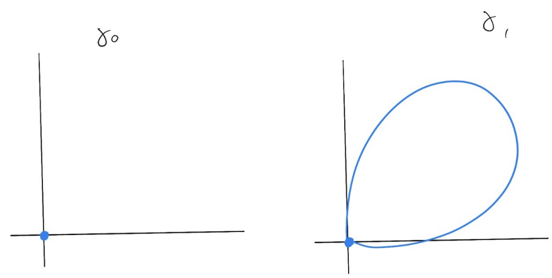


Figure 16: γ_0 and γ_1 represent the same loop up to homotopy

We pretty much calculated $\pi_1(\mathbb{R}^2, 0)$, and that's just the trivial group because every loop is homotopic.

That's a nice set, but he doesn't like sets that much. He claims that π_1 has more structure—it's the fundamental group, so surely it must be a group as well!

Well, what is a group? It's not part of the Ross curriculum, so we should write it down.

Group

A group is a set G with a binary operation $*$ (so we write it $(G, *)$) where

- (associativity) $\forall a \, \forall b \, \forall c, (a * b) * c = a * (b * c)$,
- (identity) $\exists e, \forall a, e * a = a * e = a$,
- (invertibility) $\forall a, \exists a', a * a' = a' * a = e$.

Jenny

[Wrote on iPad.] Is Aareyan's shirt backwards?

Well, what operation can we do to two loops?

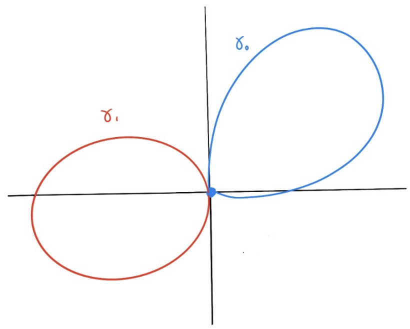


Figure 17: Two loops

We can certainly do one of them and then the other one. How do we write this down?

Well, we need to define that piecewise, where we traverse the first one and then the second one. So,

$$\gamma_0 \star \gamma_1(t) = \begin{cases} \gamma_0(2t) & 0 \le t \le \frac{1}{2}, \\ \gamma_1(2t-1) & \frac{1}{2} \le t \le 1. \end{cases}$$

Note that the speed doesn't matter since we can just continuously slow it down or speed it up, and that will be a homotopy. Also, at $t = 1/2$ itself, even though it looks like we've given multiple definitions for $(\gamma_0 \star \gamma_1)(1/2)$, they are actually the same because of our assumption that both are loops, so $\gamma_0(1) = \gamma_1(0)$.

Aareyan

Thanks for the answer. Is your name Ziyao?

What

What? (Someone said "it's Lucas.")

Aareyan

Oh, so you are the other Lucas.

Well, associativity starts to get scary. For instance, $a * (b * c)$ does the first loop a in half the time, while $(a * b) * c$ does the first one in a fourth of the time. However, since we've defined this up to homotopy, we can convince ourselves that they are basically the same because speed doesn't matter.

Also, what is the inverse of a loop?

Just go backwards! That gets us back to the trivial loop under homotopy because we can just unravel the loop.

Let's work out an example. Well, let's just consider $\pi_1(\mathbb{R}^2)$.

That's weird notation because we're used to considering $\pi(\mathbb{R}^2, x_0)$ with a point of reference, but this really does not matter. No matter what base point we choose, the resulting groups will be isomorphic because of path-connectedness.

But, if we really wanted to be formal, we would consider $\pi(\mathbb{R}^2, 0)$. But, anyways, that is just $\{e\}$ where e is the trivial loop—every loop is homotopic to the trivial loop as we showed earlier.

3.2 Circle

What about $\pi_1(S^1)$? That's not going to be the trivial group.

Andrew

So, we are going backwards, right? Just like your shirt?

Aareyan

Let's blame someone who made me get no sleep yesterday by crashing out for three hours in my room.

Well, what about the path that goes around the circle exactly once? That's not the same as the trivial loop since we have a hole in the middle. So, this is not the trivial group.

Instead, we claim this is $\mathbb{Z}$ under addition, where the number of times we go around (say, clockwise or counterclockwise—it doesn't matter).

Let's sort of lift up the diagram to understand the loop better. That gives us a helix that goes above the circle in a sense.

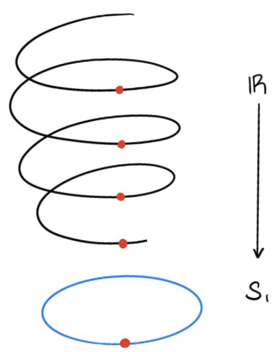


Figure 18: $[0, 1]$ mapped onto a "spiral staircase"

Specifically, we are just mapping $[0, 1] \to \mathbb{R}$ so as to stretch out the path and make it easier to work with. That map gives the z-coordinate, while we stay at the same x- and y-coordinates. Basically, we have the points on this helix are $(\cos(2\pi s), \sin(2\pi s), s)$.

The idea is that the staircase will represent the total progression along the circle. Sure, we might backtrack repeatedly on the actual path, but that will just be represented by retracing part of the spiral staircase. As a result, if we look at just the shape that's traced out, it'll sort of erase all the backtracking.

And, this will be unrigorous, but Aareyan claims that finding such a map is not too hard. If $\gamma\colon [0, 1] \to S^1$ is our original map, let's call that $\tilde{\gamma}\colon [0, 1] \to \mathbb{R}$, and we want it to satisfy the property that $\gamma = p \circ \tilde{\gamma}$ in the sense that p projects the height into the actual x- and y-coordinates on S^1.

That is, $p(s) = (\cos(2\pi is), \sin(2\pi is))$, and that's what gives the staircase its *helix* shape.

There are some other properties about this explosion upward into a helix, but those are left to the reader.

Well, the homotopy up there is quite nice. Earlier, we said that this erases the backtracking, and that's because any two paths are homotopic in this new space. We can do convex combinations because there's no hole.

Specifically, a convex combination is just a weighted average where we change the weights. For instance, when we earlier talked about how every loop of $\mathbb{R}^2$ is homotopic to the trivial loop, we just took a varying weighted average of the original loop and the trivial loop, which gave us a continuous function and proved homotopy.

In particular, convex combinations only works when the space in between the two loops has no holes in it, i.e., convex spaces. That is, the set between them is convex. As a result, so long as nothing odd is going on, any two loops will be homotopic (e.g., by taking $s\gamma_1 + (1 - s)\gamma_0$). This, by the way, came up in Prof. Holder's class, in which convex combinations are how we define simplices. In fact, simplices will appear again.

In particular, we claim that the homotopy class depends only on the integers, i.e., how many integer points appear. Recalling that we earlier noted that backtracking is essentially erased in this new representation, we only need to think about the integer points because that describes how many times we go around in total.

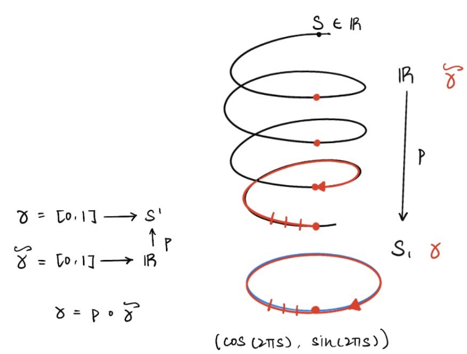


Figure 19: Path in $[0, 1]$ vs. path in lifted staircase

But, since we only consider the integer points, then that's how we get the group being isomorphic to $(\mathbb{Z}, +)$.

Here's a recap of the proof.

- First, we lift up the path so that we get a much nicer path. The helix is purely for visualization purposes, and we could just consider taking the loop to a line segment that erases the data in the x- and y-coordinates. This function is $\tilde{\gamma}$.
- Then, we only consider the integer points. All we care about is the number of integer points since that's precisely the number of times we wind around the circle.

Random comment I found interesting—how do we know that $\tilde{\gamma}$ doesn't go on infinitely? That's just because, if it did, then it would oscillate super super fast and not be continuous.

By the way, if we go the other way (i.e., we choose a direction at the start and we go in the opposite direction), then that would be interpreted as going down instead of up.

Well, we don't need to be too rigorous, and our intuition should carry us throughout.

3.3 Functoriality

Let's talk more about the fundamental group. One thing is its *functoriality*, which just means that the fundamental group (a group-theoretic concept) plays nicely with homeomorphisms (a topological concept). If you search up this term online, you'll hear about functors, which are just like meta-functions that take different categories (e.g., groups and topological spaces) to each other. That's the formal way to encode that the different objects are related to each other.

In particular, functorality would show that the fundamental group is *invariant*, so it will stay the same under homeomorphism.

Well, we claim that if f is a continuous function $(\underline{X}, x_0) \to (\underline{Y}, y_0)$ such that $f(x_0) = y_0$, then it induces a homomorphism $f_*: \pi_1(\underline{X}, x_0) \to \pi_1(\underline{Y}, y_0)$.

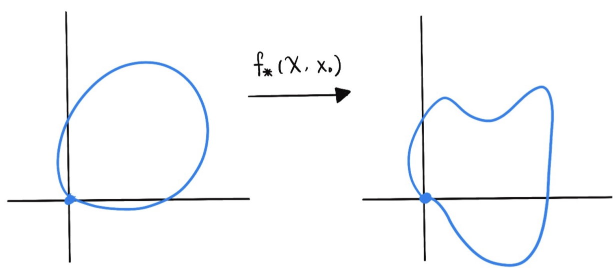


Figure 20: Homomorphism f_*

Aareyan

What is a homomorphism?

BT

Family 13! I taught you this!

Family 13

We only learned automorphisms!

JCs

Audible laughs.

Well, the general idea is that it preserves the algebraic structures. So, one thing is that $f_*(ab) = f_*(a)f_*(b)$.

By the way, that automatically shows that $f_*(e) = e_*$ (the identity element of the second group) because $f_*(e) = f_*(e)f_*(e)$, so multiplying by the inverse $f_*(e)^{-1}$ gives $f_*(e) = e_*$. That also shows that inverses are preserved.

Anyhow, how can we prove that here? The idea is just to set $(f_*(\gamma))(t)$ to be $f(\gamma(t))$. This seems like a perfectly reasonable definition—just apply the homomorphism to each point on the loop at each time step, we do need to prove this weird thing called being well-defined.

If we have had experience with equivalence classes, then we will know that just defining an operation can be difficult. For instance, we can't define exponents in modular arithmetic because, in general, the results will differ even if they map to the same thing.

As a result, we need to show that homotopic loops give homotopic loops.

And then, it's not too hard to show that it preserves the operation by showing the homotopy explicitly using the definitions. Well, anyway, let's go further.

Claim.

Let $f : \underline{\overline{X}} \to \underline{\overline{Y}}$ be a homeomorphism. Then, $f_* : \pi_1(\underline{\overline{X}}, x_0) \to \pi_1(\underline{\overline{Y}}, y_0)$ is a group isomorphism (where we define x_0 and y_0 as before).

Well, we showed part of that because an isomorphism is just a bijective homomorphism. Further, we know that the first function is a bijection (since it's a homeomorphism), which implies we can sort of describe the inverse of the second one using that of the first. This is the idea of functoriality showing up again.

So, that's why the fundamental group is super hard in general. What about higher-dimensional spheres?

Aareyan

In fact, Connor will describe that right now.

Connor

Wait, right now? Let's take a 5-minute break, actually.

Connor

[9 minutes later.] It's lock-in o'clock!

4 Scarier Holes

What Aareyan showed us for the past hour was finding a fundamental group and some examples—both the circle and $\mathbb{R}^2$.

Doing it for the circle was very hard. What if we have a *really* scary object? The spiral staircase method won't work then.

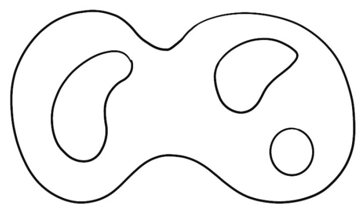


Figure 21: *Really* scary object

We can try another direction.

4.1 Way #1 (Products)

Instead, let's see how we can build the fundamental group if we understand smaller topological spaces.

A topological space is just a nice-enough subset of $\mathbb{R}^n$.

Specifically, let's say that $\underline{\overline{X}}$ and $\underline{\overline{Y}}$ are both topological spaces. We can take their product, which is just

$$\underline{\overline{X}} \times \underline{\overline{Y}} = \{(x, y) \mid x \in \underline{\overline{X}}, y \in \underline{\overline{Y}}\}.$$

For instance, the square is just $[0, 1] \times [0, 1]$.

What about $S^1 \times S^1$?

Connor claims this is a torus, but here's how we see that.

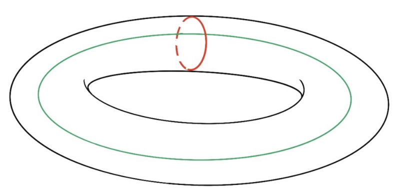


Figure 22: $S^1 \times S^1$

Here's another way to see that. First, the plain $[0, 1] \times [0, 1]$ is a boring square.

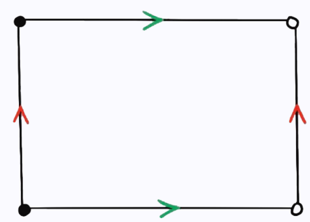


Figure 23: $[0, 1] \times [0, 1]$

But, S^1 is basically the same as $[0, 1]$, except $0 = 1$. Basically, we identify the two endpoints (i.e., treat them the same). But, when we multiply by the other set, then this identification stretches out the endpoints in a sense, so the the two *edges* are the same.

In particular, we fold the two together. The technical term for that is identifying them together, which means we treat the two edges as the same in classic Pac-Man style.

Topologists do something weird where the exponent is the dimension. So, T^2 is the torus in two dimensions (in general, a torus is just something with one hole), and we write $S^1 \times S^1 \cong T^2$. A torus is two-dimensional because we just consider the surface.

So, S^2 is not $S^1 \times S^1$ but rather the two-dimensional sphere. In fact, $S^1 \times S^1 \ncong S^2$ because the former has too many holes.

Well, is it true that if we know the fundamental group of each factor, then we also know the fundamental group of the product? Yes! (But, this will be a bit more complicated for unions, which is later.)

First, any path $\gamma\colon [0, 1] \to \underline{X} \times \underline{Y}$ is sort of like $\gamma(t) = (\gamma_x(t), \gamma_y(t))$. Basically, we've decomposed things, and paths in $\underline{X} \times \underline{Y}$ are just pairs of paths where one is on $\underline{X}$ and one is in $\underline{Y}$. We could consider this a sort of parameterization of the path, but it is distinct from a regular parameterization because $\underline{X}$ and $\underline{Y}$ don't need to be one-dimensional.

Moreover, a homotopy in this case is just a pair of homotopies in each component. That is, $H(t, s) = (H_x(t, s), H_y(t, s))$ where we have a separate homotopy in $\underline{X}$ and in $\underline{Y}$.

So, that's a nice correspondence.

Connor

That's a terrible brace, but still better than Aareyan's.

Connor

There's a fly that just flew (yes, flies fly). Oh! Now it's on my glasses!

Thor

That's the same fly that's been on me!

That's the start of a correspondence, but let's show that the algebraic structures are the same, as well.

As a set, $\pi_1(\overline{\underline{X}} \times \overline{\underline{Y}}, (x_0, y_0))$ is just $\{(\gamma_x, \gamma_y) \mid \gamma_x \in \pi_1(\overline{\underline{X}}), \gamma_2 \in \pi_2(\overline{\underline{Y}})\}$.

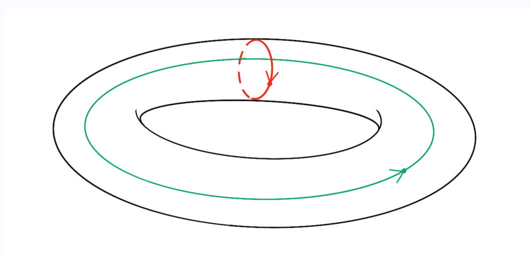
> **Sarah**
>
> Max, can I have some popcorn?

So, at least as a set, the fundamental group of the product is the product of the fundamental groups. We just get the same coordinate-by-coordinate. Specifically, $(\gamma_x^0, \gamma_y^0) * (\gamma_x^1, \gamma_y^1) = (\gamma_x^0 * \gamma_x^1, \gamma_y^0 * \gamma_y^1)$.

> Yes, we are using superscripts for indexing, which really annoys people in Operations Research for some reason.

But anyway, we see that the isomorphism is not just as sets, but actually as a group isomorphism. Cool!

So, the idea is just to work in $\overline{\underline{X}}$ individually and separately in $\overline{\underline{Y}}$. This is a really powerful theorem, and that immediately allows us to compute the fundamental group of a torus.

> Well, we have $T^2 \simeq S^1 \times S^1$, so that means that $\pi_1(T^2) = \{(\pi_x, \pi_y) \mid \pi_x \in \pi_1(S^1), \pi_y \in \pi_1(S^1)\}$.
>
> We're just breaking them down using our theorem from before. (Also, we didn't write $\pi_x, \pi_y \in \pi_1(S^1)$ since technically the S^1 are the same space but different in the sense that they do not interact).
>
> Anyways, let's now apply Aareyan's work that the fundamental group of S^1 is $\mathbb{Z}$. So, we take the product of $\mathbb{Z}$ with itself, and we get $\mathbb{Z}^2$. That's just an integer lattice, and we can think of the two options as the two ways of going around the torus, as follows.
>
> 
>
> **Figure 24:** Two loops generators on a torus
>
> Hence, we can think of this as points in the integer plane where addition is just adding the components separately.

By the way, $\overline{\underline{X}} \times \overline{\underline{Y}}$ is just the same as $\overline{\underline{Y}} \times \overline{\underline{X}}$, not necessarily as the same set, but at least homeomorphic where the homeomorphism is interchanging the two.

4.2 Way #2 (Unions)

This is just the union. While products were a bit notationally heavy, unions are not so simple. The fundamental group is definitely not just going to be the union of the original two.

By the way, union is a bit scary to think about. For instance, it would make no sense to take the union of things in different dimensions, so we'll consider taking the union only in a fixed one. For instance, if two things are subsets of $\mathbb{R}^2$, then their union is, as well.

Well, the scary part is that the union depends on how exactly we take their union. For instance, if the holes overlap, then the result might have a different number of holes than otherwise.

So, things depend on the intersection, and that itself depends on orientation and position in space.

For instance, here are two subsets $\overline{\underline{X}}$ and $\overline{\underline{Y}}$ where some of their holes just happen to perfectly match up. While it's true that there are other cases where the holes could disappear, we illustrate another interesting possibility.

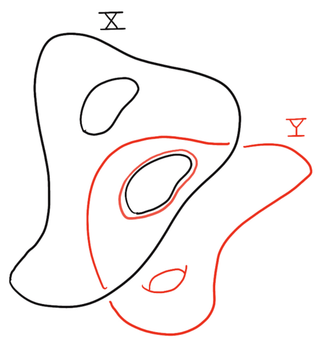


Figure 25: $\overline{\underline{X}}$ and $\overline{\underline{Y}}$ with overlapping hole

Meanwhile, there are some cases where the two holes do not overlap. The above and below shapes are quite literally the same (homeomorphic), but one has four holes and the other has three holes.

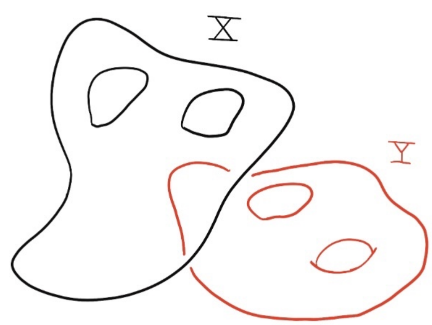


Figure 26: $\overline{\underline{X}}$ and $\overline{\underline{Y}}$ with no shared hole

> **Everyone**
>
> Applauds as Max kills a bug with Expo spray.

> **Sarah**
>
> Can I have some more popcorn?

> **Max**
>
> Guys, the June bug is crunchy, but the moth was not.

> **Connor**
>
> It's lock-in o'clock. Sǔo Dìng (锁定)!

Connor wants to reiterate why the intersection matters. In the first case, the union could have three holes because one hole was counted twice. But, in the second case, there was no double-counting, and the two holes were counted separately.

That makes things a lot more complicated. So, we'll start off simple and build up to a general result.

Well, let's consider the union of two circles at a single point. Well, π_1(two circles joined at a point) isn't even that hard.

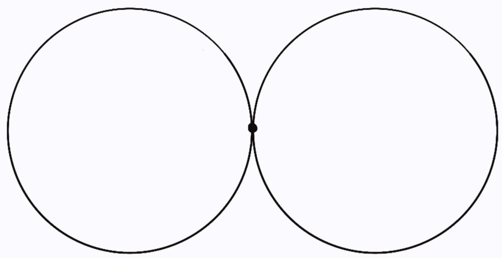


Figure 27: Kissing circles

Specifically, we can just switch back and forth to get loops in this union.

In that sense, they are quite independent of each other. In fact, if we go around the first circle, then the second, then backwards for the second circle, then that is not the same as just going around the second circle.

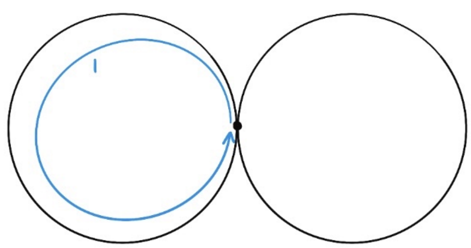


Figure 28: A first loop going around the left circle

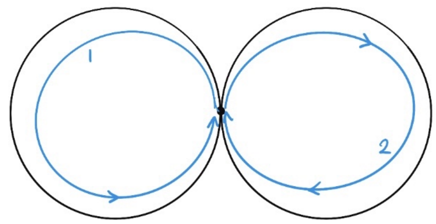


Figure 29: A second loop going around the right circle

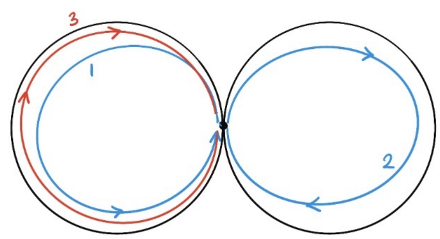


Figure 30: A third loop going *backward* around the left circle

If we only did the first and third loops, they would cancel out to give us the trivial loop. But, since we insert the second loop going around the right circle, then there's no longer any interaction. That is, the identities do not extend across elements from the other group, and this makes the group *free*. It is also a lot more complicated than a regular product.

> Algebraically, suppose we have two copies of the integers $\overset{1}{\mathbb{Z}}$ and $\overset{2}{\mathbb{Z}}$, so $\pi_1(S^1_1) = \overset{1}{\mathbb{Z}}$ and $\pi_1(S^1_2) = \overset{2}{\mathbb{Z}}$.
>
> And then, those are generated by $\overset{1}{1}$ and $\overset{2}{1}$. This is very strange notation.

Well, the general concept of a free group is that it is not commutative to the fullest extent. Specifically, it satisfies the fewest algebraic relations possible. So, if we have something like $\overset{1}{1} * \overset{2}{1} * (\overset{1}{-1})$, this is not the same as $\overset{1}{1} * (\overset{1}{-1}) * \overset{2}{1}$ (that latter one simplifies to $\overset{2}{1}$). Even though $\overset{1}{1}$ and $\overset{1}{-1}$ are inverses and satisfy an algebraic relation, that relation is interrupted by $\overset{2}{1}$ and can't pass through what is in the other group.

But, here's what a free group is rigorously.

> **Free Group**
>
> If G and H are groups, the **free group** is $G \star H = \{$"words" made with elements of G and $H\}$.

In other words, the two groups are independent and don't interact at all, so they are basically free from relations.

> That is, the only time we can simplify is if they are directly consecutive—there is nothing on the other side interfering with it.

So, this is sort of what we do with unions. Here's a theorem. Below, when we say connected, we just talk about the intuitive notion with nice topological spaces (where path-connectedness and connectedness and everything else are actually the same).

Weak Seifert-van Kampen

If $\overline{\underline{X}}$ and $\overline{\underline{Y}}$ are spaces and $\overline{\underline{X}} \cap \overline{\underline{Y}}$ is connected and $\pi_1(\overline{\underline{X}} \cap \overline{\underline{Y}})$ is trivial (is the trivial group, so $\mathbb{Z}_1$ with only one element), then $\pi_1(\overline{\underline{X}} \cup \overline{\underline{Y}}) = \pi_1(\overline{\underline{X}}) \star \pi_1(\overline{\underline{Y}})$ aside from some technicalities.

For instance, we do need other conditions, like the intersection being open. But, it works in most cases that we care about.

For instance, in this case with two kissing circles, the intersection is just a point, so its fundamental group is trivial. That means that the fundamental group of the pair of kissing circles is $\pi_1(\text{kissing circles}) = \mathbb{Z} \star \mathbb{Z}$.

Well, here's an application of that to $\pi_1(\mathbb{R}^2 \setminus \{x_1, x_2\})$. Those x_1 and x_2 are just arbitrary distinct points. We also call this guy the twice-punctured plane.

Well, let's split this up, so $\overline{\underline{X}}, \overline{\underline{Y}} \subset \mathbb{R}^2$ both contain one hole. We also want their intersection to contain neither hole.

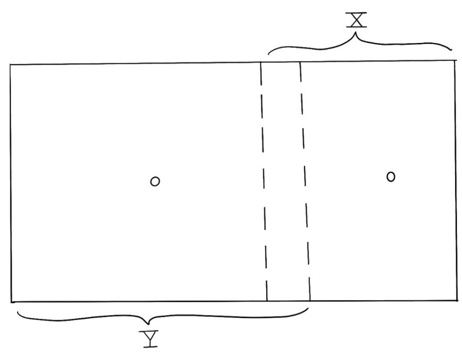


Figure 31: $\pi_1(\mathbb{R}^2 \setminus \{x_1, x_2\})$

It's important that this intersection is open (we don't contain the boundary), and that's why it's homeomorphic to $\mathbb{R}^2$ by just expanding it (we could use the tangent function if we wanted). Further, this intersection is quite obviously connected.

This means that our condition is satisfied. But, what is the fundamental group of either?

We can do the staircase/spiral argument on each component. That's a hand-wavy explanation, but it's basically the circle (since the hole is the same in both cases).

By the way, a deformation retract is another method we might perform. It's a map that still preserves the fundamental group (and many other properties) but isn't as restrictive as a homeomorphism. Then, the once-punctured plane actually turns this into a circle, and we would be done.

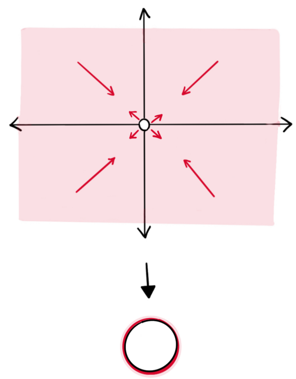


Figure 32: The plane contracts to a circle surrounding the hole

Either way, we have $\pi_1(\overline{X} \cup \overline{Y}) \cong \pi_1(S^1) \star \pi_1(S^1) \cong \mathbb{Z} \star \mathbb{Z}$.

Scary Unions

Well, what if $\pi_1(\overline{X} \cap \overline{Y})$ is not trivial? Let's say there's a hole in the intersection.

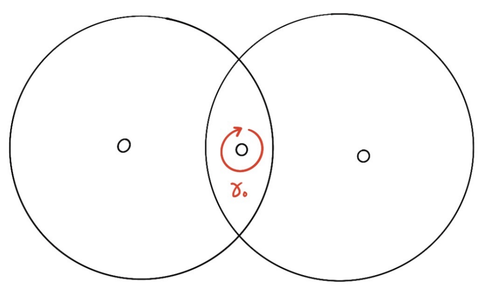


Figure 33: Hole in $\pi_1(\overline{X} \cap \overline{Y})$

In the free product, we assumed that we can do a little bit of $\overline{X}$ and do a little bit of $\overline{Y}$, and they don't interact. But here, it's possible to have both a (non-trivial) loop in $\overline{X}$ and a loop in $\overline{Y}$.

So, our free product would fail because we would think of them as different when they are really the same. Instead, we need to define an amalgamated product (a very fun word) that is sort of a free group except we specify that the common loops really are the same in some formal way. The normal way to do this is modding out by an equivalence relation, and that's precisely what we'll do.

Don't worry if you haven't seen quotient groups. It's just like $\mathbb{Z}_{17}$, where we identify some things using an equivalence relation. That is, two things are equivalent if they differ by a multiple of 17. This is a very hand-wavy notion, but as long as you understand the idea of smushing a whole class of things into one variable, then you get what a quotient group is.

Here, two things are equivalent if *the only difference* is that we use different versions of the loops in the intersection.

Amalgamated Product

Let G_1, G_2, and H be groups. Then, given the homomorphisms $\varphi_1 : H \to G_1$ and $\varphi_2 : H \to G_2$, we define the **amalgamated product** as

$$G_1 \star_H G_2 = G_1 \star G_2 \Big/ \{\varphi_1(\gamma_0) = \varphi_2(\gamma_0) \mid \gamma_0 \in H\}.$$

So, in the sense that $\mathbb{Z}_{17}$ can be thought of as a bunch of *equivalence classes* rather than actual numbers, the same goes here—we take the free product and then say that some things are the same.

For now, φ_1 and φ_2 are just given, but the real statement of Seifert-van Kampen will tell us what homomorphisms to use.

It's a quirk that we use $\star_H$ to denote the "intersection," but, even though there are many options for φ_1 and φ_2 that could give wildly different results, they are left implicit and are not captured by the notation.

In particular, H would represent the fundamental group of the intersection $\overline{X} \cap \overline{Y}$, while G_1 and G_2 are the fundamental groups of either $\overline{X}$ or $\overline{Y}$. Then, since the intersection is *basically contained* within both $\overline{X}$ and $\overline{Y}$, then φ_1 and φ_2 will send loops in the intersection to the equivalent loop in $\overline{X}$ and $\overline{Y}$, respectively.

That is, no matter if we consider the thing as part of $\pi_1(\overline{X})$ or $\pi_1(\overline{Y})$, it's the same element/loop.

For instance, let's say we have $g_1 \in G_1$, $k_1 \in G_1$, $g_2 \in G_2$, $k_2 \in G_2$, and where $k_1 = \varphi_1(h)$ and $k_2 = \varphi_2(h)$.

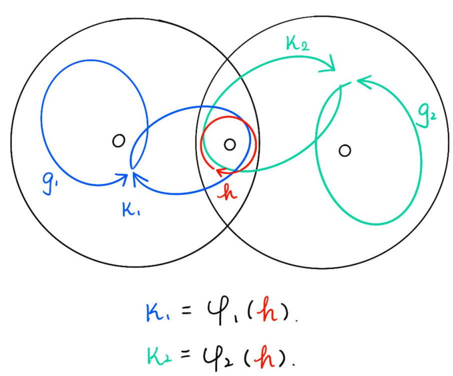


Figure 34: $\pi_1(\overline{X} \cup \overline{Y})$ where

Then, consider $g_1 * k_2 * k_1^{-1} * g_2$. In the free product, this would not simplify. But, this should intuitively simplify because k_1 and k_2 aren't actually different loops, and they should cancel out.

Algebraically, the reason this actually does simplify is because of our homomorphism. Since $\varphi_1(h) = \varphi_2(h)$, then

$$g_1 * \varphi_2(h) * \varphi_2(h)^{-1} * g_2 = g_1 * g_2.$$

In other words, we've just identified them to be the same. That's the general amalgamated product, but the following theorem tells us exactly how to choose our homomorphisms.

Seifert-Van Kampen

If $\overline{X}$ and $\overline{Y}$ are spaces with $\overline{X} \cap \overline{Y}$ connected, then there exist $\varphi_x : \pi_1(\overline{X} \cap \overline{Y}) \to \pi_1(\overline{X})$ and $\varphi_y : \pi_1(\overline{X} \cap \overline{Y}) \to \pi_1(\overline{Y})$ that is induced by functoriality.

Functoriality is just a fancy term encoding the idea of groups and topological spaces playing nicely with each other. In a sense, we are thinking of h as already being in our topological space, and the fundamental group should reflect that. So, these homomorphisms aren't really doing anything that interesting.

The only reason we use them is that we want to be rigorous. For instance, it would be wrong to say $\mathbb{Z} \subset \mathbb{R}$ because each real number is probably defined with Dedekind cuts (sets) or something. Instead, we need to say that $\mathbb{Z}$ is isomorphic to a subring of $\mathbb{R}$ (there is a homomorphism from $\mathbb{Z}$ to $\mathbb{R}$), and that's sort of what we're doing here. We're choosing the canonical one here.

So, the amalgamated product isn't too easy to work with because it just formalizes something we already knew. It doesn't give us something explicit— there are still things to check, and even if we know everything about the groups we had from before, then there's still some work to do for the amalgamated product.

Is the weak Siefert-Van Kampen theorem an immediate consequence of this version? Yeah! That's because we are just quotienting by something trivial (like, $\mathbb{Z}_0$ is just $\mathbb{Z}$).

5 Interlude — Applications

It's *lock-in o'clock*. Nope, here's a joke.

Dr. K (Neil Kolekar)

Why did the plane crash in a zigzag?

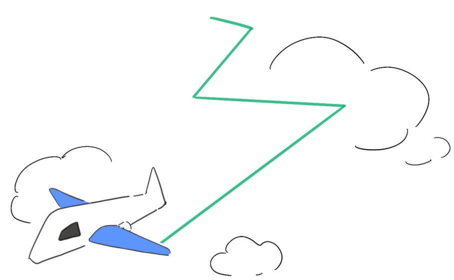


Figure 35: Dr. K's plane crash

Answer

Because the plane had one puncture, and the fundamental group of a once-punctured plane is $\mathbb{Z}$.

Connor

Why are the counselors yapping so loudly?

5.1 Fundamental Group of the Sphere

We'll do one more fundamental group, and that one will be of the sphere. This is S^2 because it's two-dimensional, so we want $\pi_1(S^2)$.

> We will use Seifert-van Kampen and compute the amalgamated product.

Maybe we do two hemispheres. Then, $\underline{\overline{X}}$ would be the upper half and a bit, while $\underline{\overline{Y}}$ would be the lower half and a bit.

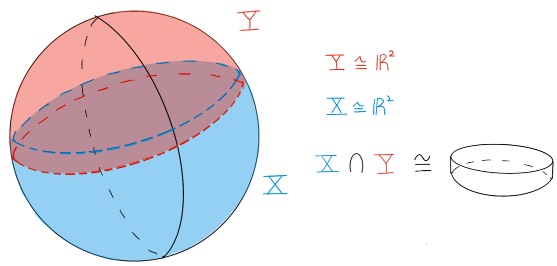


Figure 36: $\pi_1(S^2)$ as "hemispheres"

> There's a bit of overlap in the middle. For instance, if Connor's head is a sphere, then we can make $\underline{\overline{X}}$ be everything strictly above his mouth and $\underline{\overline{Y}}$ be everything strictly below his nose.
>
> Importantly, his mouth is below his nose, so there's an overlap.

Well, if we take the top half of a sphere and uncurl it, then both $\underline{\overline{X}}$ and $\underline{\overline{Y}}$ will be homeomorphic to $\mathbb{R}^2$ (because they are open spaces).

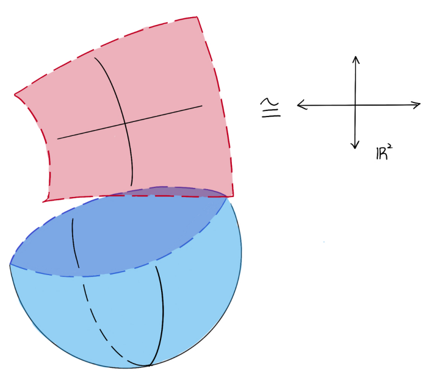


Figure 37: Unfolding the upper hemisphere

Well, the intersection of $\underline{\overline{X}} \cap \underline{\overline{Y}}$ is just a cylinder, so its fundamental group is $\mathbb{Z}$ because cylinders are just circles. In particular, cylinders are $S^1 \times [0, 1]$, and the latter has a trivial fundamental group, so we just use our theorem on products.

> In particular, $\pi_1(S^2) \cong \pi_1(\underline{\overline{X}}) \star_{\pi_1(\underline{\overline{X}} \cap \underline{\overline{Y}})} \pi_1(\underline{\overline{Y}})$. But, $\pi_1(\underline{\overline{X}}) = \pi_1(\underline{\overline{Y}}) = \pi_1(\mathbb{R}^2)$, and all are just the trivial group!

So, we get $\{e\} \star_{\pi_1(\text{cylinder})} \{e\}$. Well, the free product is trivial, but the amalgamated group is even simpler than the free product (since we identify even more things).

Mustafa

Hey, "what is" is trademarked! *[After Connor asked "what is $\{e\} \star \{e\}$?"]*

So, $\pi_1(S^2) = \{e\}$. That's because $\pi_1(\underline{X})$ and $\pi_1(\underline{Y})$ are both useless, so we don't even care about what their intersection is.

Well, we think we got off lightly because we computed this quite easily. But, this is actually bad news. So far, the fundamental group has helped us understand holes.

But, $\pi_1(S^2)$ is trivial even though it does have a hole! In a sense, this is a sign that higher-dimensional holes aren't being captured by π_1. It only sees lower-dimensional holes.

5.2 Distinguishing Spaces

But, before we move on from π_1, let's find an application to show that $\mathbb{R}^2$ and $\mathbb{R}^3$ are not the same (sort of as promised by Matthias).

The idea is to distinguish $\mathbb{R}^2 \setminus \{0\}$ and $\mathbb{R}^3 \setminus \{0\}$? Well, $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{Z}$.

But, $\pi_1(\mathbb{R}^3 \setminus \{0\})$ is trivial, and we can see that in many different ways. We could do the same Seifert-van Kampen method, but we could also project any loop onto the sphere (with some special cases for going through 0) and show the homotopy because $\pi_1(S^3)$ is trivial.

It's sort of like a deformation retract like we talked about before, though we still need to consider the special case of the origin.

It seems super trivial to show that $\mathbb{R}^2$ and $\mathbb{R}^3$ are different, but this is really the only method for understanding such spaces, and we can't do any better. Yes, it's obvious, but this really is the only way to show it rigorously. Fundamental groups are the only way we can work with things since we haven't really defined topological invariants.

Aareyan

Let's work out... Let's work out... Let's work out!

Kaka

Do your 11 push-ups right now.

Timothy

There's a Planet Fitness near the movie theatre, by the way.

Well, it was not quite satisfying because this seems obvious. Why are we doing so much work to prove something intuitive? Here's a nice theorem that is not at all obvious.

5.3 Fixed Points

By the way, $\mathbb{D}^2$ is the disk in two-dimensions, which includes its boundary and interior (whereas, $\mathbb{D}^1$ is a line segment). It is two-dimensional because we need two coordinates—even though S^1 and $\mathbb{D}^2$ don't seem too different and are embedded in the same dimension, they are considered to have different dimensions as shapes.

Brouwer's Fixed Point Theorem

Let $f : \mathbb{D}^2 \to \mathbb{D}^2$ be continuous and bijective. Then, it has a fixed point, so there exists some x such that $f(x) = x$.

This is a nice and elegant proof.

Proof.

Well, suppose $f : \mathbb{D}^2 \to \mathbb{D}^2$ is a map without a fixed point. Let's start by considering some function g that maps $\mathbb{D}^2$ to S^1 by taking the ray from $f(x)$ to x and intersecting with the boundary. This requires the assumption that $f(x) \neq x$ (if they are the same, then there's no well-defined ray). We can also convince ourselves that this is a continuous map.

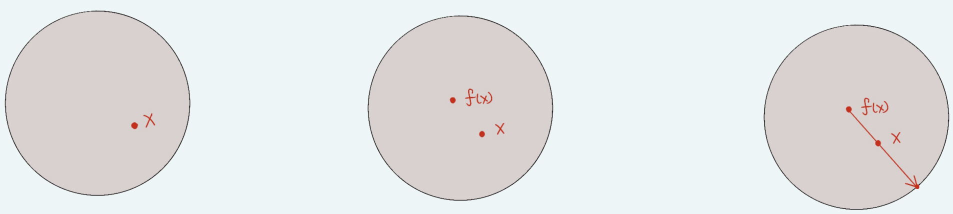


Figure 38: Points x, $f(x)$, and the projection onto S^1

One thing we might notice is that $g|_{S^1} = \mathrm{id}_{S^1}$. That is, if we only consider the boundary points (so, we restrict to S^1), then g does nothing—if we are already on the circle, then projecting to the circle does nothing! (The formal term for this is a retraction mapping.)

Well, *this is absurd*, and we claim this produces a contradiction. (It seems crazy that we could tear apart a disk and retract it onto the circle while being continuous. For instance, we definitely couldn't do this for $\mathbb{D}^1$, the line segment, because there's clearly no continuous map that sends everything to the endpoints.)

But, the way we think about this is again through the topological invariants. It turns out that the fundamental group can only get smaller under retractions (which is a general property). Let's start by defining $g_* : \pi_1(\mathbb{D}^2) \to \pi_1(S^1)$ as follows. For any loop γ, then $g(\gamma)$ is also a loop—where we just apply g on each point. This makes sense as a function because g gives something in S^1.

So then, when we mean that the retraction can only shrink the fundamental group, we mean that g_* is a surjection. In particular, is it true that given any loop on the surface, we can find a loop in the disk that maps to it? Yeah, and it's *easy*! The loop on the circle is still a loop in the disk. But, that is really funny though, because we've found a surjection from the trivial group $\pi_1(\mathbb{D}^2)$ to $\pi_1(S^1) \cong \mathbb{Z}$—even though $\mathbb{Z}$ is a whole lot bigger.

Alternatively, we've shown that $\pi_1(S^1)$ is actually just the trivial group itself. That's because any loop in $\mathbb{D}^2$ is homotopic to the trivial loop.

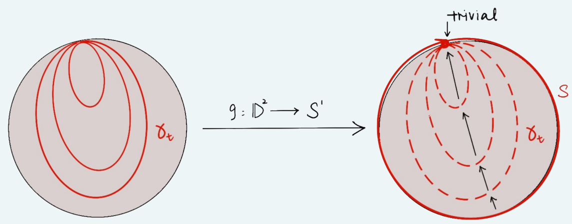


Figure 39: Homotopy of loops in $\mathbb{D}^2$

But anyway, that contradicts our assumption at the beginning that $f(x) \neq x$ for all x, which was used to justify the retraction. In particular, it implies that $f(x) = x$ for some x, which intuitively solves our problem because we can still rip apart our disk into a circle under the retraction map as long as some points get killed. □

We used the Brouwer Fixed-Point Theorem a lot in applied math. We often need to do math to show that our method makes sense. In fact, we discussed it in operations research, but we won't be using it later today—it's just an application. In fact, there are tons of fixed-point theorems, like the recursion theorem about a computer reading its own code.

Hopefully, this shows that algebraic topology is not just obvious stuff.

5.4 Fundamental Theorem of Algebra

Let's do another one, which is a bit more involved and pretty weird (why is algebra showing up?).

Fundamental Theorem of Algebra

Every nonconstant polynomial in $\mathbb{C}$ has at least one root in $\mathbb{C}$.

Oliver Lippard

JCs, did you get your Fundamental Theorem of Algebra wrong? *[Timothy — "Every polynomial in $\mathbb{C}$ has no roots in $\mathbb{C}$."]*

Kaka

Lock in! One moment I am not reading the document, and this happens.

Proof. Well, suppose by way of contradiction that p is a polynomial in $\mathbb{C}[x]$ with no complex roots in $\mathbb{C}$. Well, for each $r \in \mathbb{R}^+$, define $f_r : [0,1] \to \mathbb{C}$ via

$$s \mapsto \frac{p(re^{2\pi is})/p(r)}{|p(re^{2\pi is})/p(r)|},$$

which is going to be something on the unit circle since we are just dividing by the norm (the divisions are legal since, by hypothesis, p has no roots in $\mathbb{C}$). Further, just by smoothly varying r, we get a homotopy between f_r and f_0, which we can think of as paths that takes s as an input. By the way, we write "are homotopic to each other" as $\sim$, so $f_r \sim f_0$ for each r.

But, anyways, how do we bring in topology? Well, we'll notice that f_0 is just the trivial loop that sticks everything at 1 because

$$\frac{p(0 \cdot e^{2\pi is})}{p(0)} = \left|\frac{p(0 \cdot e^{2\pi is})}{p(0)}\right| = 1$$

for all s. In fact, f_r is always a loop because $e^{2\pi i \cdot 1} = 1$, so we get back to where we started.

So, if we could show that f_r is not homeomorphic to the trivial loop, then we would be done! In fact, this seems doable. All of these are paths on S^1, and we know what the fundamental group is on that surface. In particular, the nontrivial loops are just the ones that wind around the circle in some direction, so it makes sense that we should get something like that.

We want to find some value of r so that f_r just loops around the unit circle. How might we make that?

Well, this would be easy if $p(z)$ were literally just z^n or something. Then, a ton of things cancel and all that falls out is $e^{2\pi ins}$ (the algebraic details are left to you). And, that definitely loops around the circle—n times, in fact.

That being said, we don't need $p(z)$ to be precisely z^n. It could have other terms, but as long as those other terms are tiny, then we would still get the same winding behavior. In particular, under certain conditions, we might be able to say that $f_r(s)$ is homotopic to the loop traced out by z^n.

So, let's start by defining $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$. We're assuming that our function is monic (has a leading coefficient of 1)—otherwise, we could just divide. Well, $p(z)$ might be a bit messy, but perhaps we could show that it's homotopic to another loop where, excluding the leading term, everything in the polynomial vanishes. Specifically, define $p_t(z)$ to be $z^n + t(a_{n-1}z^{n-1} + \cdots + a_0)$ where $t \in [0, 1]$.

We're done, right? We see that $p_0(z)$ is homotopic to $p_1(z)$ (it gives a continuous deformation to a plain winding), and that means they also define homotopic loops. But, not quite. All we know is that $p(z) = p_1(z)$ has no complex roots. But, what if $p_0(z)$ does have complex roots (it definitely does)? So, let's be rigorous—that is salvageable.

In a similar vein as before, define $f_{r,t} : [0, 1] \longrightarrow \mathbb{C}$ via

$$s \mapsto \frac{p_t(re^{2\pi is})/p_t(r)}{|p_t(re^{2\pi is})/p_t(r)|}.$$

Sure, we it might not follow that p_t has no roots, but since we can play with r, we are going to show that there is some r for which p_t has no roots with magnitude r. Then, even if p_t does have roots, that'll be fine because they'll all be small.

Well, we could use the fact that there are always going to be finitely many roots and go from there. But, that presupposes too much knowledge. Instead, let's construct one. In particular, pick some huge r that is both greater than 1 and greater than the sum of all the coefficients in absolute value, i.e., $r > \max\{|a_{n+1}| + \cdots + |a_0|, 1\}$. Then, we claim that p_t cannot possibly have a root whose magnitude is exactly r.

Well, the original proof was by contradiction, but that's ugly. Let's consider what it means for $p_t(z) = 0$ when $|z| = r$. Well, it means that $z^n = -t(a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0)$. Taking norms tells us that $|z|^n = t|a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0|$. The magnitudes are nice because we can swap $|z|$ with r to get $r^n = t|a_{n-1}r^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0|$. Next, the exponent of r is at most $n - 1$, but since $r > 1$, we get $r^n \leq t|a_{n-1}r^{n-1} + a_{n-2}r^{n-1} + \cdots + a_0r^{n-1}|$.

We now perform two steps at once: taking out r^{n-1} and using the triangle inequality equality to break apart the absolute values (which states that $|a + b| \leq |a| + |b|$ sort of because of a triangle). That gives

$$r \leq t(|a_{n-1}| + |a_{n-2}| + \cdots + |a_0|).$$

But, $t \leq 1$, so this actually gives $r \leq |a_{n-1}| + |a_{n-2}| + \cdots + |a_0|$. That's absurd by definition.

That is, if we are big enough and choose a big enough r, then the leading term will be much bigger than everything else—there cannot be arbitrarily large roots, in a sense. As a result, $f_{r,t}$ is a homotopy from the winding that we get from z^n to f_r. Moreover, since f_r is homotopic to the trivial loop f_0, and because homotopy is transitive, then looping n times is homotopic to not looping. That's absurd by what we just proved. □

Yay! Let's do one last application.

5.5 Topological Groups

Let's consider the notion of topological groups. Hopefully, we should know about groups by now.

Topological Surface

A **topological group** is a group with a topological structure where both the group operation and group inverse are continuous.

What are some examples?

Well, the real number line under addition is a topological space with a group structure. Clearly, addition and subtraction are continuous. The circle (the unit circle in complex numbers under multiplication, i.e., rotation) is also a group.

More generally, $(\mathbb{C}^\times, \cdot)$ is another topological group, where we remove zero.

What about the torus? It's a topological group!

Well, remember that $T^2 \cong S^1 \times S^1$, and we can just add in S^1. It's just pointwise addition (if we are thinking of S^1 as $\mathbb{R}/\mathbb{Z}$) or pointwise multiplication (if we are thinking of S^1 as the complex numbers).

Here's a scarier example!

Consider a two-holed torus (like what we did in Lam's class).

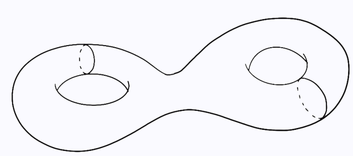


Figure 40: Two-holed torus

Well, the two-holed torus comes from the octagon. (In general, you can make an n-holed torus with a $4n$-gon where we connect $aba^{-1}b^{-1}cdc^{-1}d^{-1}\cdots$. The inverse just means that we flip it when we read it out, sort of like how in a square, we flip the second edge of a to point in the same direction as before. That's called a commutator, or something.)

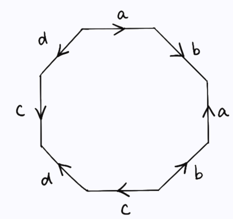


Figure 41: Gluing an octagon into a two-holed torus

But, that doesn't tessellate the plane, so just adding doesn't help.

Fake! There is actually no group structure, and we'll prove this.

Connor

[Draws the two-holed torus T^2.] This is called getting **mogged**.

Roughly, our proof will begin with the result that if a group has two operations that are compatible with each other in a sense, then the group must be abelian (everything commutes). This is known as the Eckmann-Hilton argument (in fact, for any set with two operations that both have an identity and satisfy some other condition, then those two operations are the same and are both commutative and associative).

We'll demonstrate that argument in general, but we'll keep the notation of fundamental groups just so that we can easily apply things later. This argument will assume that G is itself a group and, then, use that to show that $\pi_1(G)$ has two binary operations that play nicely with each other.

Proof.

First, there is the regular operation on $\pi_1(G)$ just by concatenating loops, but there's another one $m_*\colon \pi_1(G) \times \pi_1(G) \to \pi_1(G)$ using the group operation on G itself. Specifically, $m_*(\gamma, \gamma')(x) = \gamma(x)\gamma'(x)$ where we use the natural multiplication on G to multiply $\gamma(x)\gamma'(x)$. (This new operation might not have anything to do with the usual operation of concatenation, but actually this argument is sufficient to show they are.)

It's interesting that $\pi_1(G) \times \pi_1(G) \cong \pi_1(G \times G)$ by what Connor proved earlier about fundamental groups of products, but that won't be significant.

Then, the identities of each map are going to be the same (where identities are just if we treat these maps as operations, really). In fact, they must be the trivial loop in either cases, and that's not too hard to show. However, we want a more general proof because we will later use this argument in a different context. In order to extend this argument to other cases, let's just assume that these maps satisfy the *interchange* property that $m_*(a, b) \circ m_*(c, d) = m_*(a \circ c, b \circ d)$.

By the way, Aareyan says unit instead of identity, which doesn't make sense because everything is a unit (and has an inverse).

In this case, showing the interchange property is pretty easy by our piecwise definition. But now, in order to show that this operation is commutative, let's switch to an alternative notation instead of writing out so many $\circ$'s and m_*'s.

Well, we can represent the property from above, $m_*(a, b) \circ m_*(c, d) = m_*(a \circ c, b \circ d)$, with numbers in a matrix. For instance,

$$\begin{matrix} a & b \\ c & d \end{matrix}$$

could be thought of as four numbers that could be combined in two options. Here, if we think of the horizontal direction as m_* and the vertical direction as $\circ$ (the usual operation in $\pi_1(G)$), then we can think of grouping the four numbers in two ways as

$$\begin{pmatrix} a & b \end{pmatrix} \atop \begin{pmatrix} c & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a \\ c \end{pmatrix}\begin{pmatrix} b \\ d \end{pmatrix}.$$

By the identity, the grouping doesn't matter since we get the same result either way, meaning that we can just ignore the brackets and work like that.

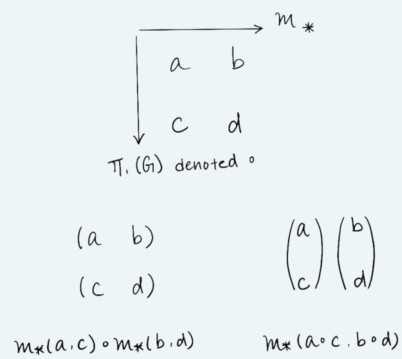


Figure 42: Two multiplications on a, b, c, and d

So now, this manipulation becomes pretty simple. For instance, to see that the identities are the same in general, let's plug in $a = d = e$ and $c = b = e_*$. Here, e is the identity of m and e_* is the identity of m_*. Then, we get

$$\begin{pmatrix} e & e_* \\ e_* & e \end{pmatrix} = \begin{cases} e \circ e = e, \\ m_*(e_*, e_*) = e_*. \end{cases}$$

That verifies that the identity is the same, so let's just call it e. Consider substituting $b = c = e$. Doing the multiplication shows that

$$m_*(a, b) = \begin{matrix} a & e \\ e & b \end{matrix} = \begin{matrix} e & a \\ b & e \end{matrix} = m_*(b, a).$$

A similar argument shows that $a \circ b = b \circ a$, and that's really what we want. (In fact, this demonstrates that $\circ$ and m_* must be the same because that matrix simplifies to both.) But anyway, this shows that in every topological group, multiplication must be commutative.

So, the upshot of topological groups is that they're abelian, which agrees with our experience. But, what is the fundamental group of the two-holed torus?

However, another exercise for the reader shows that multiplication is not commutative in the two-holed torus, which means it cannot be a topological group. In particular, we (Aareyan) really want(s) to show that it is non-commutative in order to show that it is not a topological group. Briefly, as we saw in Figure 41, this group is generated by the edge identifications a, b, c, and d subject to the relation that $aba^{-1}b^{-1}cdc^{-1}d^{-1} = e$. In particular, this comes because if we go across each edge in order, then we'll get back to where we start. This completely characterizes our group.

If you are familiar, we would write this in the abbreviated notation $\langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} = e \rangle$ or just $\langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle$ if we get lazy and imply that it's equal to the identity. This is called the group presentation, with the generators on one side and the relations on the other.

Crucially, it seems intuitive that this is not commutative because it's way too complicated. (The way you would prove that is to find a surjective homomorphism to a group that you definitely know is not commutative.) Hence, by applying the argument above, it can't be a topological group. □

We have now explored the fundamental group quite thoroughly. Time for homology.

6 Homology

What is homology? That's a shockingly difficult question to answer. We'll discuss many things in homology and then build up to singular homology, which is the most important.

But, for now, we'll talk about simplicial homology.

6.1 Simplicial Homology

The idea is going back to talking about notions of holes. Remember that $\pi_1(S^2)$ is trivial even though S^2 has a hole. So, homology is sort of another idea in capturing holes. Here are some shapes. For instance, the triangle below is a planar graph that lives on some sort of surface, and we could formalize that notion through properties of the surface. But, even just thinking about the triangle as its own entity, we can sort of talk about it as having a hole.

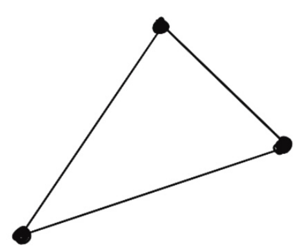


Figure 43: A 1-dimensional triangle, not filled in the middle

Some of these have holes and some don't. Also, the holes are different dimensions. But, the idea is that these shapes are simple enough to work with rigorously. That being said, we will work with more complicated and possibly higher-dimensional structures.

Simplex

An n-dimensional **simplex** has $n + 1$ vertices in general position.

These are the generalizations of a triangle.

- A 0-dimensional simplex is a point (with $0 + 1 = 1$ vertex).

Figure 44: The 0-dimensional simplex, $\triangle^0$

- A 1-dimensional simplex is a line segment (with $1 + 1 = 2$ vertices).

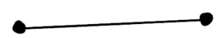


Figure 45: The 1-dimensional simplex, $\triangle^1$

- A 2-dimensional simplex is a triangle (with $2 + 1 = 3$ vertices).

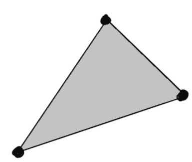


Figure 46: The 2-dimensional simplex, $\triangle^2$

- A 3-dimensional simplex is a tetrahedron (with $3 + 1 = 4$ vertices).

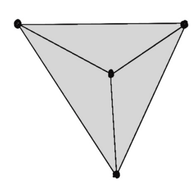


Figure 47: The 3-dimensional simplex, $\triangle^3$

General position just means that they aren't degenerate. For instance, the three vertices of a triangle are not allowed to lie on the same line. As a result, the simplices are the simplest possible polytope in a given dimension.

But, simplices are pretty boring. We can easily figure out what holes they have. So, let's instead discuss some more complicated, higher-dimensional structures.

Simplicial Complex

Let's start with simplices, where an n-dimensional simplex has $n + 1$ vertices. But then, a **simplicial complex** is a bunch of simplices connected.

There are more restrictions, of course. For instance, simplices are not allowed to intersect except at their boundary (what is a boundary?). But, just think of the most natural triangulations, in a sense.

Simplices and simplicial complexes are very simple ideas. These are finite and only have finite information, and they are sort of an approximation that is easier to work with.

Well, let's consider if we fill in the simplices. So, we have triangles surrounding a pentagon, for instance. Then, we can't fill in the pentagon because it's not a simplex.

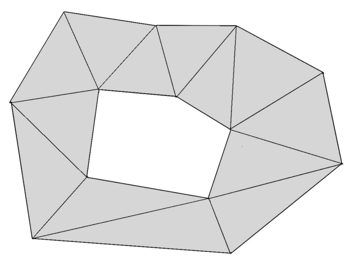


Figure 48: Pentagon surrounded by triangles (two-simplices)

And here is the same structure with the boundaries shown.

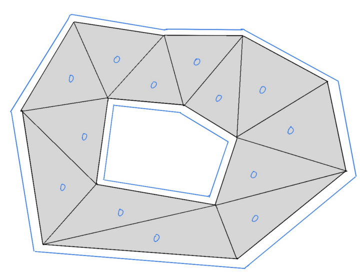


Figure 49: Boundaries on the above shape

The hole is cool.

Well, we can study boundaries in order to get a grasp on holes. The idea is that the boundary of a boundary is empty. That's a general principle in geometry—the boundary of a ball is the sphere, and the sphere has no boundary. (The boundary actually has an intricate definition, but that will be later.)

But, is it true that everything with no boundary is itself a boundary? This seems like the start of a new detection technique for higher-dimensional holes. Basically, surrounding the hole is a path, but it's not the boundary of anything in the sense that it will never be the complete boundary. It is part of the boundary, but you cannot get it on its own as a boundary of something.

So, here's that more rigorously.

Chain

An n-**chain** is a collection of n-simplices (n-simplices, i.e., the definition, not n simplices).

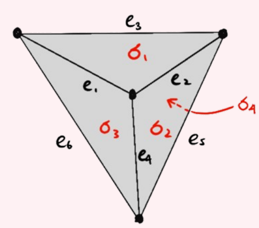


Figure 50: The *surface* of a tetrahedron is a 2-chain (collection of triangles)

Boundary Map

We love functions. An n-**boundary map** takes n-chains to $(n-1)$-chains. So, its domain is the set of n-chains and its codomain is the $(n-1)$-chain.

In our case, it basically takes two-dimensional simplices to their boundaries, which are one-dimensional simplices. It's a one-chain.

Cycle

A n-**cycle** is an n-chain with no boundary.

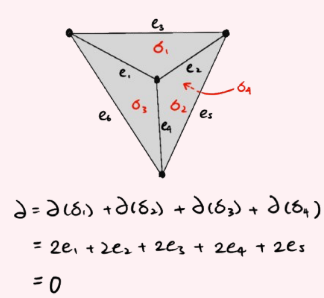


Figure 51: Edges cancel in the surface of tetrahedron, giving a boundary of 0

For instance, 0-cycles are vertices.

Boundary

An n-**boundary map** is any n-chain that is in the image (under the boundary map) of an $(n+1)$-chain.

These four definitions are very important. Our definition of a hole is basically an n-cycle that is not an n-boundary. In a sense, it has no boundary, but it isn't the boundary of anything. And, that doesn't seem to depend on dimension.

> **Aareyan**
>
> It's *co* bad! *[In reference to cohomology, which would be a general version of simplicial homology but with* `co-` *in front of everything.]*

Here, we sort of see two holes—the inside one-chain and the outside one-chain. But, in the formal definition, these two will correspond to the same hole as we shall see.

> Can we get back to topology and think about groups and stuff and not just sets?

Can we add chains? That's a weird idea, but this would turn chains into a group. Specifically, it's just addition modulo 2, so it's just all the points that are included an odd number of times.

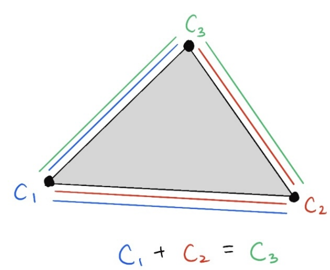


Figure 52: The edge in both C_1 and C_2 is canceled

> The reason for that definition is to make it a group (if we make it the union, that wouldn't work).

The identity is the empty chain, while inverses are just themselves (a pre-Coxeter group).

Also, we'll denote the n-dimensional boundary with ∂_n (which is the same symbol as for partial derivatives, weirdly).

> For instance, let's draw one edge of a 1-simplex (a normal triangle) and call it e_1, but it's also a 1-chain. So then, if we call the vertices v_1, v_2, and v_3, with v_3 opposite e_1, we have
>
> $$\partial_1(e_1) = v_1 + v_2,$$
> $$\partial_1(e_2) = v_1 + v_3,$$
> $$\partial_1(e_3) = v_2 + v_3.$$
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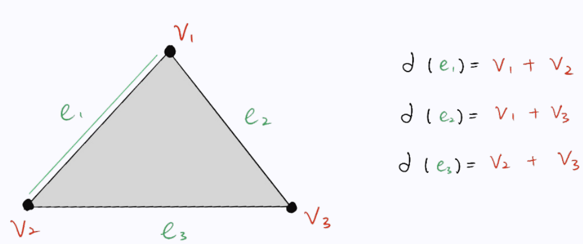
> 
>
>
> **Figure 53:** Vertices and edges of a normal triangle

What is $\partial_1(e_1 + e_2)$? Well, the boundary is just $v_2 + v_3$ geometrically, but it turns out that we can formalize this by treating them as symbols. Well, that would give us $\partial_1(e_1 + e_2) = 2v_1 + v_2 + v_3$, but the idea is that we modulo by two. That's because including a boundary twice sort of cancels it out.

As another example, if we have a triangle, then the vertices are not boundaries because each vertex is included twice. You can also think about this as the `XOR` operation.

We're starting to see that the boundary map is a group homomorphism. Indeed, we have a bunch of groups and a bunch of group homomorphisms between them. We'll arrange this into what we call a **chain complex** (that's the same word *complex*, but they sadly mean different things). That might seem similar to vector stuff, at least for Thor.

$$0 \longleftarrow C_0(\underline{X}) \xleftarrow{\partial_1} C_1(\underline{X}) \xleftarrow{\partial_2} C_2(\underline{X}) \xleftarrow{\partial_3} C_3(\underline{X}) \longleftarrow \cdots$$

Here, $C_n(\underline{X})$ is the group of chains under the operation from above.

Well, we can visualize this with a chain of boundary maps where we reduce the dimension each time. When we arrange the chains in that way, it encodes all the relevant information in a neat way. Further, there's implicit information. If we take the boundary and then the boundary of that, we get 0 (where we denote the trivial group as 0). So, $\partial_i \circ \partial_{i+1}(x) = 0$ for all x and i.

By the way, that is the algebraic meaning of the boundary of the boundary being the empty set. Later on, we'll define things called images and kernels, so we can say that the image of ∂_i is contained within the kernel of ∂_{i+1} for each i.

This is basically the definition. In fact, this idea shows up repeatedly and is not limited to algebraic topology, but it does get its name from that.

Well, let's go further. The group of n-cycles is the set $Z_n(\underline{X}) \subseteq C_n(\underline{X})$ defined by $\{z \in C_n(\underline{X}) \mid \partial_n(z) = 0\}$.

That is, Z_n contains the chains that get sent to zero under the boundary map. We use Z because C has already been taken by "chain." Anyways, $Z_n(\underline{X})$ remains a group because the boundary map is a homomorphism—it preserves structure.

Meanwhile, the group of n-boundaries is the set $B_n(\underline{X}) \subseteq C_n(\underline{X})$ defined as

$$\{b \in C_n(\underline{X}) \mid \exists\, c \in C_{n+1}(\underline{X}), \partial_{n+1}(c) = b\}.$$

Finally, we can state our definition of a homology. Remember, the homology will try to describe holes.

Recall that we wanted our notion of holes to be n-cycles that were not n-boundaries. We can achieve this with group quotients.

Homology

The n^{th} homology group $H_n(\underline{X})$, i.e., the group of n-dimensional holes, is defined as $Z_n(\underline{X}) \big/ B_n(\underline{X})$.

This is a quotient group where we work with things in $Z_n(\underline{X})$ but where they are the same up to adding things in $B_n(\underline{X})$.

Let's do an example. We can start with the 1-simplices surrounding a hole. This hole is considered a one-dimensional hole since it can be found in $H_1(\underline{X})$. So, $\underline{X}$ is the interior of the hole. In particular, $Z_1(\underline{X}) = \{0, e_1 + e_2 + e_3\}$. What is

$B_1(\underline{\overline{X}})$? Well, there's literally nothing to take the boundary of, so it has to be $\{0\}$. As a result, we have

$$H_1(\underline{\overline{X}}) = \{0, e_1 + e_2, e_3\} \Big/ \{0\}.$$

Well, quotienting out by the trivial group is stupid because we're saying two things are equal if we can add a multiple of 0 to get one from the other. Hence, we still have $H_1(\underline{\overline{X}}) \cong \{0, e_1 + e_2 + e_3\} \cong \mathbb{Z}_2$.

We've detected a hole because we have a homology group that is non-trivial! In particular, because $\mathbb{Z}_2$ is raised to the first power, then there is one hole.

Let's compute more homology groups for this triangle.

Max

Timothy has transformed into Darth Vader. *[After he put on his hood. Totally not preparing for the cartel.]*

Well, that's $H_1(\underline{\overline{X}})$, while $H_2(\underline{\overline{X}}) = 0$.

What about $H_0(\underline{\overline{X}})$? Well, $Z_0(\underline{\overline{X}})$ is the set of things sent to 0, but everything is sent to 0. So, $Z_0(\underline{\overline{X}})$ is just $C_0(\underline{\overline{X}})$. That gives

$$\{0, v_1, v_2, v_3, v_1 + v_2, v_1 + v_3, v_2 + v_3, v_1 + v_2 + v_3\} \Big/ \{0, v_1 + v_2, v_2 + v_3, v_1 + v_3\}.$$

Meanwhile, the only boundaries in $B_0(\underline{\overline{X}})$ can be vertices. Specifically, we have two vertices at a time, so we have $B_0(\underline{\overline{X}}) = \{0, v_1 + v_2, v_1 + v_3, v_2 + v_3\}$.

Then, modding out gives $H_0(\underline{\overline{X}}) \cong \mathbb{Z}_2$.

Does this mean that we have a 0-dimensional hole?

Nah. It turns out that $H_0(\underline{\overline{X}})$ represents the number of connected components, and it turns out you can prove it.

By the way, the idea is that we'll always have $\mathbb{Z}_2^{\text{something}}$ (even for $H_0(\underline{\overline{X}})$) because adding anything to itself always gives 0.

More generally, $\mathbb{Z}_2^n$ means the number of holes. That is, the structure of $H_0(\underline{\overline{X}})$ determines the number of holes, but if we wanted to actually find where the holes are (e.g., what simplices bound them), then we can look at the generators in some sense.

Let's now consider what happens when we instead consider $\underline{Y}$, which is the filled-in triangle (a two-simplex surrounded by one-simplices).

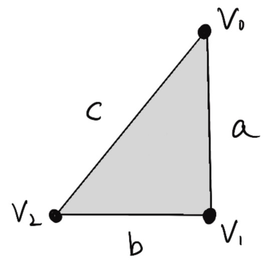


Figure 54: Simplex $\underline{\overline{Y}}$

Well,

$$H_0(\underline{\overline{Y}}) \cong \mathbb{Z}_2^1,$$
$$H_1(\underline{\overline{Y}}) \cong \mathbb{Z}_2^0,$$
$$H_1(\underline{\overline{Y}}) \cong 0 = \mathbb{Z}_2^0.$$

This idea is typically referred to as simplicial homology with $\mathbb{Z}_2$ coefficients, while Aareyan will discuss simplicial cohomology with $\mathbb{Z}$ coefficients. Matthias lied. We will never get to cohomology.

And, we'll later get to real homology for topological spaces instead of just these weird combinatorial things.

Let's work out some examples. But also, let's consider what would happen if we used different shapes (none of this involved the topology of a triangle). And, would they give the same homology?

Those are all homeomorphic to a circle, like a square, for instance.

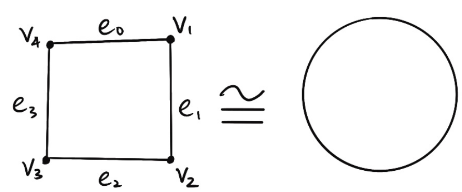


Figure 55: Square homeomorphic to a circle

That gives us this chain complex.

$$0 \longleftarrow C_0(\underline{\overline{X}}) \xleftarrow{\partial_1} C_1(\underline{\overline{X}}) \longleftarrow 0$$

$$\langle v_0, v_1, v_2, v_3 \rangle \qquad \langle e_0, e_1, e_2, e_3 \rangle$$

Note that the $\langle\rangle$ notation is just saying "generated by." It's taking all the sums of the generators instead of just listing out everything. That's much more efficient. Another note on notation—we usually write ∂_1, but Aareyan got lazy and just wrote ∂ because there's only one non-trivial boundary map in this case.

By the way, we can write explicitly

$$\begin{aligned}\partial e_0 &= v_0 + v_1,\\ \partial e_1 &= v_1 + v_2,\\ \partial e_2 &= v_2 + v_3,\\ \partial e_3 &= v_3 + v_0.\end{aligned}$$

So, we get $H_0(\underline{X}) = \langle v_0, v_1, v_2, v_3 \rangle \big/ \langle v_1 = v_2 = v_3 = v_0 \rangle = \mathbb{Z}_2^1$.

What is $H_1(\underline{X})$?

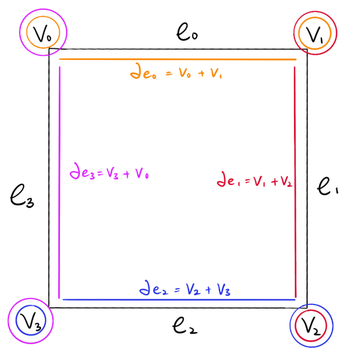


Figure 56: Since each vertex is included twice, the sum is 0 mod 2

Well, what boundaries go to zero? You can convince yourself that the only combination is $e_0 + e_1 + e_2 + e_3$.

By the way, this should give you a hint that the simplicial complex doesn't matter.

Aareyan

At the end of the day, all that matters are the topological spaces and the friends we made along the way.

Let's now work out a higher-dimensional example (the other ones for this guy are boring—either 0 or just $H_0(\underline{X})$, which is $\mathbb{Z}_2^1$). In particular, let's just look at the torus. There's only one vertex because we glue them all together, so we get v. And, there are two natural edges, a and b, and a diagonal edge c just for creating simplices.

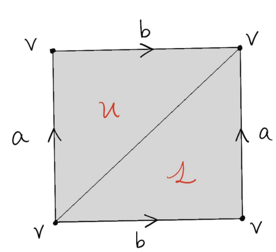


Figure 57: A torus unfolded as a square

But, we only really need to look at one of the triangles, an upper and a lower one.

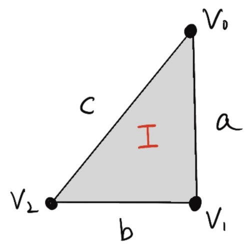


Figure 58: Half of the above square

Well, here's the exact sequence that we need to work with.

$$0 \longleftarrow C_0(\overline{\underline{X}}) \xleftarrow{\partial_1} C_1(\overline{\underline{X}}) \xleftarrow{\partial_2} C_2(\overline{\underline{X}}) \longleftarrow 0$$

$$\langle v \rangle \qquad \langle a, b, c \rangle \qquad \langle U, L \rangle$$

Let's start with computing some things.

$$\begin{aligned} \partial_1(a) &= 0, \\ \partial_1(b) &= 0, \\ \partial_1(c) &= 0, \\ Z_1(\overline{\underline{X}}) &= \langle a, b, c \rangle, \\ B_0(\overline{\underline{X}}) &= 0. \end{aligned}$$

Everything is a cycle. Nice! And, nothing is a boundary. What about ∂_2?

$$\begin{aligned} \partial_2(U) &= a + b + c, \\ \partial_2(L) &= a + b + c, \\ Z_2(\overline{\underline{X}}) &= \langle U + L \rangle, \\ B_1(\overline{\underline{X}}) &= \langle a + B + c \rangle. \end{aligned}$$

Let's now compute some homologies. We just have $H_0(\overline{\underline{X}}) = \langle v \rangle$ (since we are modding out by nothing), so we get $\mathbb{Z}_2^1$. Meanwhile,

$$H_1(\overline{\underline{X}}) = \langle a, b, c \rangle \big/ \langle a + b + c \rangle = \langle a, b \rangle = \mathbb{Z}_2^2.$$

So, a torus has two one-dimensional holes (the cross-section and going around the donut hole) and one two-dimensional hole (the inside). That's cool!

> **Aareyan**
>
> Why does this seem sus?

Meanwhile, $H_2(\overline{\underline{X}}) = \langle U + L \rangle = \mathbb{Z}_2^1$.

In general, the p-dimensional homology is the same as the $(n-p)$-dimensional homology (for compact spaces).

Meanwhile, the homology of a cylinder is the same as the homology of a Möbius band. It doesn't recognize that difference, and that's really heartbreaking.

Now, though, we'll start caring about orientation.

By the way, we aren't doing proofs at this time of day. For instance, we will not prove that ∂^2 gives 0.

David

Neil is playing Tetris. He is not even good.

Neil

Shut up. Lock in.

6.2 Scarier Simplicial Homology

So, simplicial homology isn't perfect. One of the uses of topology is formally distinguishing shapes; for instance, we might want to tell $\mathbb{R}^n$ and $\mathbb{R}^m$ apart when $n \neq m$ by showing they have different topological properties.

In particular, how can we distinguish a Möbius band from a cylinder? After all, the former is non-orientable (there's no sense of clockwise and counterclockwise), while the latter allows for a consistent definition. The problem here is that the negative signs disappear since we are working modulo two. But, that allows coefficients in $\mathbb{Z}_2$ to sometimes determine non-orientability because it would make sense that Möbius bands introduce negative signs while cylinders do not. *(In particular, the simplicity of $\mathbb{Z}_2$ means that twists no longer interfere with forming valid cycles over, whereas it would in $\mathbb{Z}$.)*

So, Connor and Aareyan *were* going to introduce a version of simplicial homology that uses coefficients in $\mathbb{Z}$ instead of coefficients in $\mathbb{Z}_2$, and they thought that the new version would help us distinguish the cylinder and Möbius band in terms of their simplicial homology.

But, it turns out that the homology group isn't even different. Instead, you would use the second compactly supported de Rham coefficients in order to distinguish the two.

Also, it's not that hard to show that the cylinder and the Möbius band have the same homology group. We can show that they are both homotopic to S^1, which is sort of a homeomorphism except that there's no requirement for a bijection (and, hence, a continuous inverse). It turns out that homotopy preserves the homology group for each dimension.

Besides, from the perspective of holes, they look the same anyways. We would probably say they both have a one-dimensional hole.

But, let's just go with this in order to explore how coefficients in $\mathbb{Z}$ might give us something different. *[It won't be as interesting as was originally planned, but it's still going to be a fun exercise.]* First, the cycles might be different. For instance, we might have $C_i(\boxed{X}) = \langle e_1, \ldots, e_n \rangle$, i.e., the chains generated by the edges. Then, we can have arbitrary integer combinations, so like $7e_1 - 17e_2$.

However, one thing that does change is that we alternate signs when computing the boundary of a chain. Remember, we want the boundary of a boundary ($\partial\partial$ in Aareyan's lingo) to vanish, and that only works if we have alternating signs.

We didn't have to define that when our coefficients were in $\mathbb{Z}_2$ for obvious reasons.

Connor

4:30 is not the time to define signs and stuff, so we will just say it works. Yay!

Audience

(Applauds.)

Connor

Also, we are going to Google the homology of a Möbius band instead of computing it.

Well, the Möbius band is not very interesting, but let's instead find the homology group for the Klein bottle. So, let K be the Klein bottle. It's also a non-orientable surface.

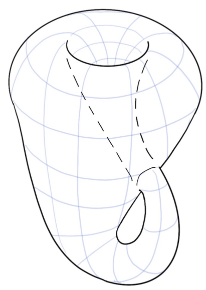


Figure 59: Klein bottle

Specifically, we have

$$H(K, \mathbb{Z}_2) = \begin{cases} H_0 = \mathbb{Z}_2, \\ H_1 = 0, \\ H_2 = \mathbb{Z}_2, \end{cases}$$

while

$$H(K, \mathbb{Z}) = \begin{cases} H_0 = \mathbb{Z}, \\ H_1 = \mathbb{Z}_2, \\ H_2 = 0. \end{cases}$$

The two choices from the two options for coefficients.

These are things that you should not try at home, but you probably shouldn't because it is impossible. Klein bottles do not embed in three-space, but it's not too hard to show. The first fold along $\mathbb{Z}$ makes it a cylinder. Normally, we fold it up to get a torus, but in a Klein bottle, we twist it first and put it inside-out. Normally, that requires self-intersection, but if we put it into four dimensions, we can avoid that self-intersection.

Aareyan

Twist it in, and then put it in.

Also, if we glue two Möbius strips, it also gives us a Klein bottle. That's another way to get it.

For non-orientable shapes, holes sort of break down. If we get things that are no longer just $\mathbb{Z}^n$, it more represents *this is the homology group* and there's not that much else we can do.

The worst thing is that the homologies are very different. But anyway, the idea is that H_1 being $\mathbb{Z}_2$ is more about non-orientability.

Kaka

You are high on tissues! *[To Mustafa.]*

Mustafa

But it smells good! Also, obviously GRH is true, but RH is false.

Recall that we've spent a long time on simplicial homology. No longer—we will be doing singular homology instead.

6.3 Singular Homology

In particular, singular homology avoids having to choose a triangulation. For singular homology, we are no longer considering just a single choice. Instead, we might as well do all of them.

Why singular if the definition is literally *not choosing a single triangulation*? Well, singular really means weird, like a *singular matrix* that is not an invertible or a *singularity* in a black hole.

Well, these are singular precisely because there is so much variety allowed with our triangulation. Literally crazy choices are allowed, like intersections in the simplex. That's crazy!

So, here's the set-up. Define an abstract simplex as the set $\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}_{\geq 0} \mid x_0 + x_1 + \cdots + x_n = 1\}$ (with nonnegative numbers). That forms an n-dimensional simplex and is denoted Δ_n. This is called Δ_n and is called an n-dimensional simplex (like an n-dimensional equilateral triangle, regular tetrahedron, etc.).

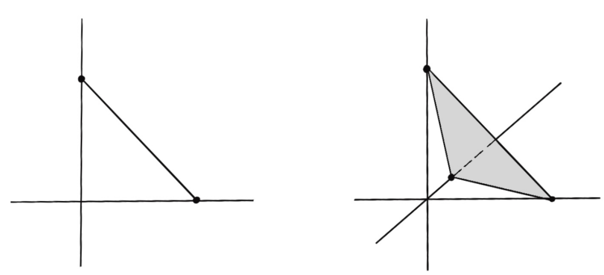


Figure 60: Δ_2 and Δ_3

There's no formal name for this construct, and "abstract simplex" is just a name we made up. Well, these are not singular or weird yet, but that's what comes next.

Well, abstract simplices are just combinatorial objects, but we build up "singular simplices" in genuine topology spaces through a map. In particular, if $\underline{\overline{X}}$ is a genuine topological space and not just a stupid graph, then an n-simplex is a map $f : \Delta_n \to \underline{\overline{X}}$. All we need is that the n-simplex is continuous. A ton of other things can be broken—there can be self-intersections and other crazy properties. But, these universal definitions make things much easier to prove.

There are very few restrictions, and we could have curved sides, for instance. There's like a really, really big number of these guys.

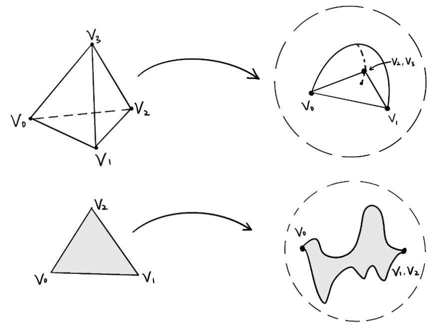


Figure 61: Maps to topological spaces of Δ^2 and Δ^3

Well, something cool is if we set $x_0 = 0$. That gives us a face that is an $(n-1)$-simplex, so we get $\Delta_{n-1} \subseteq \Delta_n$ on the i^{th} face. These faces would be like the generalized edges of the simplex. Once we know the boundaries of a simplex, then we can compute the boundaries of arbitrary chains.

But, remember that simplices are functions. So, even if we have a rigorous definition of faces in terms of sets, we need to turn those into functions. However, that's pretty easy. If we denote by F_i the i^{th} face of some simplex f, which is a set, then the boundaries of a simplex are denoted $f|_{F_i}$. The subscript restricts the domain to the i^{th} face, so it's the function that it is only defined on F_i.

In other words, boundaries of simplices make sense in the combinatorial abstract sense, but then we need to port them to $\underline{X}$. Anyways, once we have this notation, we define $\partial f = f|_{F_0} - f|_{F_1} + \cdots + (-1)^n f|_{F_n}$. That's an $(n-1)$-chain.

This is not function addition. We just have a bunch of formal symbols that don't simplify or anything—there's no interaction between them. That's sort of like how we thought of individual simplices from before as adding them.

In fact, this is sort of like the notion of a free abelian group where each simplex sort of represents a generator.

Also, no summation notation or anything at 5 AM, please.

We want to discourage the set notation, which works quite well with $\mathbb{Z}_2$, but how can you include something in a set *six times*? Or, *negative five times*? That doesn't add to our understanding.

The only reason we used set notation was because $\mathbb{Z}_2 = \{0, 1\}$, and maybe 0 represents being in the set and 1 not being in the set. At least that's sensible.

Recall the alternating sums, sort of like the sums from when we looked at coefficients in $\mathbb{Z}$ for simplicial homology (these will also be coefficients in $\mathbb{Z}$). It's not too different, except we just generalize our idea of a simplex. So, let's actually define our homology now. But, since singular homology is supposed to be the rigorous and universal counterpart to simplicial homology, let's be really formal and start talking about groups.

Well first, we have the chains. This is the group generated by all the simplices f, so we would write

$$C_n(\underline{X}) := \langle f \mid f : \Delta_n \to \underline{X} \text{ is an } n\text{-simplex in } \underline{X} \rangle.$$

Chains can be super weird with a bunch of crazy self-intersections. But, they aren't that different from just regular chains.

Again, we're just using the plus sign without truly adding them as functions. Anyways, those are our chains in n-dimensions. Remember, the chains don't need to be connected or anything.

And then, our boundary map is of the form $\partial_n\colon C_n(\underline{X}) \to C_{n-1}(\underline{X})$. We won't bother to define it explicitly, but just implicitly as the unique homomorphism that has the property that $\partial_n(f) = f|F_0 - f|F_1 + \cdots + (-1)^n f|F_n$.

That means we can define our Z_n (cycles) and B_n (boundaries) in the same way. We have literally just

$$\begin{aligned} Z_n &:= \{z \in C_n(\underline{X}) \mid \partial_n(z) = 0\}, \\ B_n &:= \{b \in C_n(\underline{X}) \mid \exists\, c \in C_{n+1}(\underline{X}), \partial_n(c) = b\}. \end{aligned}$$

Those definitions aren't really different from their counterparts in simplicial homology. Anyway, just like before, we'll define the most important of these four groups, H_n, as

$$H_n(\underline{X}) := Z_n(\underline{X}) \big/ B_n(\underline{X}).$$

Remember that we have uncountably many simplices, and we can even add them. So, C_n is massive, and so are Z_n and B_n. But, what about H_n?

Theorem 6.1

It turns out that it isn't that big. Somehow, if $\underline{X}$ is a space (subject to some technical conditions), then for any triangulation $\mathcal{T}$ of $\underline{X}$, the simplicial homology $H_n(\mathcal{T})$ is isomorphic to the underlying singular homology $H_n(\underline{X})$.

The general name for these types of theorems is a comparison theorem, so we could call this the singular-simplicial comparison theorem. There are tons of comparison theroems.

Basically, the two are the same and interchangeable. The triangulation is just like breaking our shape into a bunch of triangles.

Connor

Do any of you plan on doing probability? Put your hand down. *[In reference to the technical conditions because, statistically speaking, almost all spaces don't satisfy those nice spaces.]*

This is a little surprising at first. Both Z_n and B_n are massive. But, they are so close in size that quotienting gives something manageable (that can even be described by some combinatorial triangulation).

So, if these versions of homology are the same, then why do we even care about the two homologies?

Well, simplicial homology is better for finding the homology, but singular homology is much better for theory. No one is going to calculate a singular homology directly. On the other hand, because singular homologies are much nicer, we can use them in our proofs much more nicely.

So, here's another theorem.

Theorem 6.2

Let $\underline{X}$ and $\underline{Y}$ be topological spaces with $f\colon \underline{X} \to \underline{Y}$ being a continuous function. Then, there are homomorphisms $H_n(f)\colon H_n(\underline{X}) \to H_n(\underline{Y})$. That's in the same vein of functoriality as before.

(This has definitely happened before, but if we don't specify the nature of a function yet it goes from group to group, then it is a group homomorphism.)

It's not like we're taking the homology of a function $H_n(f)$. It's just like a notation. Other people sometimes do f_{*n}. We have to keep the n there because the dimension is important, I guess.

Well, we used an analogous fact for π_1 to show that it is invariant under homeomorphism.

So, as a corollary, if $\underline{\overline{X}} \cong \underline{\overline{Y}}$, then $H_n(\underline{\overline{X}}) \cong H_n(\underline{\overline{Y}})$.

$H^n(\underline{\overline{X}})$ is for cohomology and $H_n(\underline{\overline{X}})$ is for regular homology. Connor's muscle memory.

Here's a proof sketch, by the way, for that theorem.

Remember that for loops, we took $(f_*(\gamma))(t) = f(\gamma(t))$. Here, we're going to take g to be an n-simplex in $\underline{\overline{X}}$, but we can do something similar. We have $f \circ g\colon \Delta_n \to \underline{\overline{Y}}$. Basically, g takes things to $\underline{\overline{X}}$ and f takes things to $\underline{\overline{Y}}$. Call this composition f_n.

So, now, let's do a huge diagram.

$$\begin{array}{ccccccccc}
0 & \longleftarrow & C_0(\underline{\overline{X}}) & \xleftarrow{\partial_1} & C_1(\underline{\overline{X}}) & \xleftarrow{\partial_2} & C_2(\underline{\overline{X}}) & \longleftarrow & \cdots \\
 & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \\
0 & \longleftarrow & C_0(\underline{\overline{Y}}) & \xleftarrow[\partial_1]{} & C_1(\underline{\overline{Y}}) & \xleftarrow[\partial_2]{} & C_2(\underline{\overline{Y}}) & \longleftarrow & \cdots
\end{array}$$

Proof by I said so.

7 Sequencessssss

Too many s's.

Aareyan is extremely locked out, but let's see what we can do. So, let's talk about long exact sequences (so far, those were short exact sequences). Well, instead of looking at the whole group, maybe we just look at the holes, except for things coming from a certain part of the space. That's like looking at $\mathbb{Z}_p$ instead of $\mathbb{Z}$ as a whole. This will be some relative information.

Given some space $\underline{\overline{X}}$ and some subspace A, we might be curious about the holes in $\underline{\overline{X}}$ that are **not in** A. It is a bit more subtle, but we'll see.

$$0 \longleftarrow C_0(\underline{\overline{X}}; A) \xleftarrow{\partial_1} C_1(\underline{\overline{X}}; A) \xleftarrow{\partial_2} C_2(\underline{\overline{X}}; A) \longleftarrow \cdots$$

This isn't that hard to define. It's just
$$C_n(\underline{\overline{X}}, A) := C_n(\underline{\overline{X}}) \big/ C_n(A).$$

It turns out that the boundary maps still work, so we have a chain complex. But, we can formalize this sort of using a generalized chain complex.

Exact Sequence

It's a sequence of groups that goes as follows.

$$\cdots \longrightarrow A_{n-1} \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \longrightarrow \cdots$$

Further, $\mathrm{Ker}(f_n) = \mathrm{Im}(f_{n-1})$ for each n.

Well, we have not defined the kernel and image.

Kernel and Image

The **kernel** of a homomorphism is the things that get sent to 0, so $\mathrm{Ker}(f) := \{a \in A \mid f(a) = e\}$. The **image** of a homomorphism is the set of things that get mapped to. We have $\mathrm{Im}(f) := \{b \in B \mid \exists\, a \in A, f(a) = b\}$.

Well, we've already seen this before. For instance, here's a chain complex.

$$\cdots \longleftarrow C_{n-1}(\underline{\overline{X}}) \xleftarrow{\partial_n} C_n(\underline{\overline{X}}) \xleftarrow{\partial_{n+1}} C_{n+1}(\underline{\overline{X}}) \longleftarrow \cdots$$

And, it does indeed happen that everything in the image of ∂_{n+1} is also in the kernel of ∂_n (which is the idea that the boundary of the boundary vanishes). Alternatively, since $B_n = \mathrm{Im}(\partial_{n+1})$ and $Z_n = \mathrm{Ker}(\partial_n)$, we could view this as stating that $Z_n \supseteq B_n$ or else the quotient won't make sense.

Chain complexes are not in general the same because if $B_n = Z_n$, then our homology is trivial.

Perhaps we should now discuss exact sequences. This is why this lecture is not five minutes long.

So then, $A \overset{i}{\hookrightarrow} \underline{\overline{X}}$. Moreover, $H_n(A) \xrightarrow{H_n(i)} H_n(\underline{\overline{X}})$ (surjection). Note that $\hookrightarrow$ is for injections, and $\twoheadrightarrow$ is for surjections.

Anyways, this map between chain complexes (with a complicated connecting map) is called the zig-zag lemma. It's induced by homology because taking the homologies make this an *exact sequence* and not just chain complexes.

$$H_n(A) \xrightarrow{H_n(i)} H_n(\underline{\overline{X}}) \xrightarrow{i} H_n(\underline{\overline{X}}; A) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow H_{n-1}(\underline{\overline{X}}) \longrightarrow \cdots$$

This is sort of how we describe the information of $\underline{\overline{X}}$. Scary!

Well, how does this help? Remember all that amalgamated product business for fundamental groups? Here, we won't have that sort of thing, but instead exact sequences.

This entire thing is probably graduate-level stuff, but here's where it gets even scarier.

Here's a theorem.

Mayer-Vietoris

Let $\overline{\underline{X}} = A \cup B$ (with extremely mild assumptions on A and B). Then, there is an exact sequence (where the kernel of one map was the image of the next) as follows.

$$\begin{array}{ccccccc}
0 \longleftarrow & H_0(\overline{\underline{X}}) & \longleftarrow & H_0(A) \times H_0(B) & \longleftarrow & H_0(A \cap B) \\
 & & & & & \uparrow \\
 & H_1(\overline{\underline{X}}) & \longleftarrow & H_1(A) \times H_1(B) & \longleftarrow & H_1(A \cap B) \\
 & & & & & \uparrow \\
 & H_2(\overline{\underline{X}}) & \longleftarrow & \cdots & &
\end{array}$$

(These arrows should be zig-zaggy, but I got too lazy to make them look nice.)

Basically, we're decomposing things, and that's pretty nice. Let's do the sphere S^2 again.

Homology Group of S^2

This is the same decomposition as $\pi_1(S^2)$. Their intersection is homeomorphic to a cylinder, which is itself essentially a circle, so it has the same homology group as S^1.

Well, we start off with $H_n(\mathbb{R}^2) = \begin{cases} \mathbb{Z} & n = 0, \\ 0 & \text{else.} \end{cases}$ Also, from the circle, we know $H_n(\text{cylinder}) = \begin{cases} \mathbb{Z} & n = 0, 1, \\ 0 & \text{else.} \end{cases}$

That's not too hard to see—there are no holes in $\mathbb{R}^2$, and it's one connected component. We can do something similar for a cylinder. Well, we can redraw our diagram as follows.

$$\begin{array}{ccccccc}
0 \longleftarrow & H_0(\overline{\underline{X}}) & \longleftarrow & \mathbb{Z}^2 & \longleftarrow & \mathbb{Z} \\
 & & & & & \uparrow \\
 & H_1(\overline{\underline{X}}) & \longleftarrow & 0 & \longleftarrow & \mathbb{Z} \\
 & & & & & \uparrow \\
 & H_2(\overline{\underline{X}}) & \longleftarrow & 0 & \longleftarrow & 0 \\
 & & & & & \uparrow \\
 & H_3(\overline{\underline{X}}) & \longleftarrow & 0 & \longleftarrow & \cdots
\end{array}$$

(These should really be zig-zagged, but that looked a bit ugly when I tried doing so, and I got a bit lazy.)

Well, if something is sandwiched between two zeroes, then it must also be zero. Specifically, if we have the map $0 \xleftarrow{f} \overline{\underline{X}} \xleftarrow{g} 0$, then we know that $\mathrm{Im}(f) = \mathrm{Ker}(g)$ since we have a short exact sequence. However, $\mathrm{Ker}(g) = \overline{\underline{X}}$ because everything maps to 0 in $\overline{\underline{X}}$ (being a homomorphism). So, f is surjective, meaning that $\overline{\underline{X}}$ has only one element—and so $\overline{\underline{X}} = 0$. As a result, $H_3(\overline{\underline{X}}) = H_4(\overline{\underline{X}}) = \cdots = 0$.

However, we need to do some more work for the other ones. Let's start with $H_2(\overline{\underline{X}})$. Well, clearly, the image of 0 is going to be 0, so by exactness, the kernel of the map to $\mathbb{Z}$ is trivial, which means it must be injective. Further, because the kernel of the map from $\mathbb{Z}$ to 0 is everything in $\mathbb{Z}$ (nothing can be sent to something nonzero), so that must also be the image of the map from $H_2(\underline{\overline{X}}) \to \mathbb{Z}$. An injective surjection is a bijection, so $H_2(\underline{\overline{X}})$ is $\mathbb{Z}$ itself. Wooh!

What about $H_1(\underline{\overline{X}})$? Well, based on some higher knowledge, we know that $\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$ is actually the map $1 \mapsto (1, -1)$. That's scary. But either way, the kernel is trivial, but since that's also the image of a map from 0, then that means that $H_1(\underline{\overline{X}})$ must be 0. Wooh again!

Kaka has a future in homological algebra, which means no future.

And, what about $H_0(\underline{\overline{X}})$? We can just use the First Isomorphism Theorem to show that it must be 0. The key here is using the exactness at $\mathbb{Z}^2$. Since we know that $H_1(\underline{\overline{X}}) = 0$, we have $0 \leftarrow H_0(\underline{\overline{X}}) \leftarrow \mathbb{Z}^2 \leftarrow \mathbb{Z} \to 0$. Well, the map from $\mathbb{Z}$ to $\mathbb{Z}^2$ is given by $1 \mapsto (1, -1)$, so its image is the subgroup $\{(x, -x) \mid x \in \mathbb{Z}\}$. Therefore, by exactness at $\mathbb{Z}^2$, then the kernel of the map $\mathbb{Z}^2 \to H_0(\underline{\overline{X}})$ is also that subgroup. As a result, there is a surjection $\mathbb{Z}^2 \to H_0(\underline{\overline{X}})$. Further, since from above we know the kernel to be $\{(x, -x) \mid x \in \mathbb{Z}\}$, then we can apply the First Isomorphism Theorem. In this case, it tells us that for a surjective map, then the codomain is isomorphic to the domain modulo the kernel. So,

$$H_0(\underline{\overline{X}}) \cong \mathbb{Z}^2 \Big/ \{(x, -x) \mid x \in \mathbb{Z}\}.$$

Computing the quotient group just gives $H_0(\underline{\overline{X}}) \cong \mathbb{Z}$.

This is what a homological calculation looks like. It is hell, but at least it comes naturally once you see them enough. We have found out! We have that

$$H_n(S^2) = \begin{cases} \mathbb{Z} & n = 0, 2, \\ 0 & \text{else.} \end{cases}$$

It has one two-dimensional hole, no one-dimensional hole, and one connected component. □

The basic idea of algebraic cohomology calculations is to play two games, one with the hole and one with the image. You do them back and forth really cleverly (most of the time, the explanation isn't even done).

The math that Connor does is Galois cohomology, in which all intuition is lost—there is no connection to holes, and all they do is group theory.

Can we use this to show that $\mathbb{R}^n$ and $\mathbb{R}^m$ are different? Well, there's a generalization that shows that

$$H_n(S^m) = \begin{cases} \mathbb{Z} & n = 0, m \\ 0 & \text{else.} \end{cases}$$

Homology is our definition of a hole. Any intuitive idea about holes can be made rigorous with homology.

Let's now get to the actual proof.

It only took like ten hours to prove this.

Well, now that we have all of that, let's go.

We did define homotopy for paths, but doing it for spaces in general is not too hard. (Homotopy is different from homeomorphism—e.g., Möbius bands and cylinders are homotopic but not homeomorphic.) The second compactly supported de Rham coefficients are different between the two, and that's how we would distinguish them.

Aareyan

Suicide is not allowed at Ross.

So, we have $\mathbb{R}^n \setminus \{0\}$, and we want to show it is not congruent to $\mathbb{R}^m \setminus \{0\}$ for $n \neq m$.

When we remove a point, it's basically a sphere S^{n-1}. We can kind of like expand the hole to get a ball of a hole (like, a big hole). We are fattening the circle.

But then, the homology groups are not the same by what we just proved, and so the two spaces cannot possibly be the same. Remember that homeomorphisms are preserved under homology groups (in the sense of functoriality).

8 Cool Beans

Connor

Cool-down topics.

8.1 SUS

Well, let's say we have a circle. It's a process called SUS (short for SUSpension) that increases the dimension of some object. For instance, it can take a circle and give us a rigorou definition of a sphere, but in any dimension. That is, it will be a map $S^1 \mapsto S^2 \mapsto S^3 \mapsto \cdots$.

Remember how we computed the homology of the sphere, S^2, with an overlapping intersection of two disks? That intersection was cylindrical. Well, we can think of this instead as taking a cylinder and extending it upward in either direction, but then pinching the top and bottom to make the shape a real sphere. That's what makes this sort of a SUSpension.

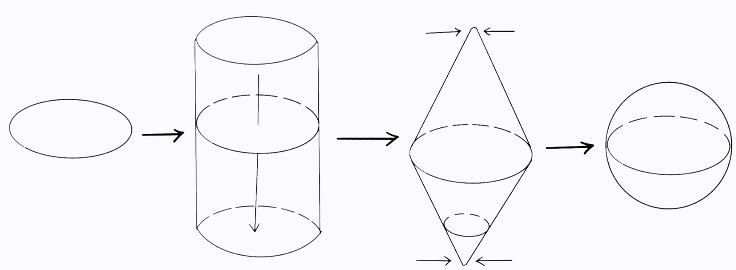


Figure 62: SUSpension of a circle

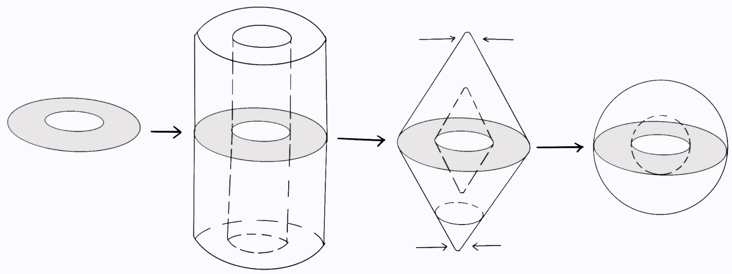


Figure 63: SUSpension of an annulus (two-dimensional ring)

But, the cylinder is basically just a circle. SUS is just a generalization of this concept.

The way we do it is by thinking of two cones over a circle (a double cone). Formally, a cone over $\underline{X}$ is just something like

$$\underline{X} \times \left([0,1] \big/ \{(x,1) \sim (x',1), \forall x, x'\} \right).$$

Basically, this is a general cone where we don't need to have S^1. We could also think of this as a generalized cylinder where the top and bottom collapse into a single point.

By the way, to define a double cone, we have $[-1,1]$ instead of $[0,1]$, and then we do some more equivalence relations to pinch the bottom of the cone down there.

Well, the idea is that the cone over S^n gives us S^{n+1}. So, the cone is like a hemisphere, and combining the two gives us something homeomorphic.

That is, $H_{n+1}(\text{SUS}(\underline{X})) = H_n(\underline{X})$ assuming $\underline{X} \neq 0$.

8.2 Homotopy Groups

We're going to explore something called higher homotopy groups and then find a relation to SUSpensions! Well, the fundamental group $\pi_1(\underline{X})$ led us to a rather interesting discussion, but how can we generalize this group? After all, what is the 1 doing?

If we are going to find a related definition, it's probably going to be by changing the domain. Specifically, instead of mapping from $[0,1]$, let's try going from $[0,1]^2$.

But, wait. Even though we have some $\gamma\colon [0,1] \to \underline{X}$, we need to say that $\gamma(0) = \gamma(1)$. What things might we need to identify in this other case?

You might say to identify the whole left edge with the whole right edge, and the whole bottom edge with the whole top edge, but that would give us a torus. Instead, we'll notice that $[0,1]$ but where $\gamma(0) = \gamma(1)$ sort of sounds like the definition of a circle—we're wrapping up the two ends.

So, instead of discussing tori, let's consider S^2.

Specifically, that gives us $\pi_2(\underline{X}, x_0) := \{\gamma\colon S^2 \to \underline{X} \text{ up to homotopy}\}$. Instead of circles, we now have two-dimensional loops, sort of like balloons (though, again, they can intersect and do all sorts of weird stuff). More generally, we can think of $\pi_n(\underline{X})$ as having n-dimensional loops.

Higher Homotopy Groups

The n^{th} higher homotopy group is defined $\pi_n(\underline{X}) := \{S^n \to \underline{X} \text{ up to homotopy}\}$.

As before, we need to fix a base point, but any base point will give us isomorphic groups.

By the way, what is $\pi_0(\underline{X})$?

Well, S^0 is just the boundary of $\mathbb{D}^1$, so it has two points. One of those points is required to be at the basepoint, while the other point can be anywhere else. Further, two paths here are homotopic if those other points are path-connected. So, π_0 is just a set of the path-connected components. For instance, if $\underline{X}$ has n distinct path-connected components, then $\pi_0(\underline{X})$ would be a set of n points.

Wait, π_0 is not a group? It's a set?

Yes—there's no obvious way to concatenate two paths—there's too few points. But, we're safe because we can concatenate loops in π_n for positive n, right? How?

I mean, there's still no obvious way to concatenate balloons. In fact, there are infinitely many ways—we could stack them horizontally, vertically, or any way we might wish. But, what did we prove earlier about a group with two operations?

The Eckmann-Hilton argument from above showed that if a group has two operations that play nicely with each other, then those are abelian. Is it possible that π_2 and higher are all abelian?

In fact, that would tell us that higher homotopy groups are almost always abelian except for π_1 when defined on topological spaces that are not topological groups. So, let's prove that π_n is an abelian group when $n \geq 2$.

Well, we can start by considering what it means to concatenate for homotopy spaces. With just S^1, there weren't too many options, but here there are actually infinitely many. The general idea when defining a concatenation is piecewise, and we could either squish vertically or horizontally.

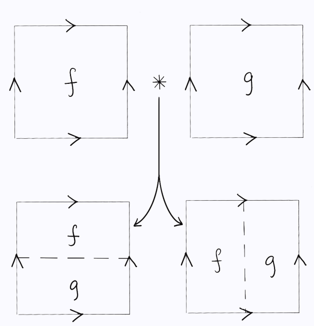


Figure 64: Different options for concatenating f and g

Well, these two options are different, but they do have the nice interchange property.

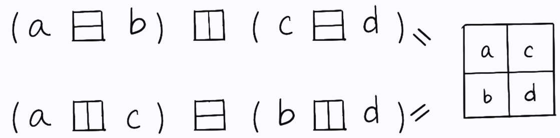


Figure 65: Interchange property

As a result, the exact argument shows that π_n is abelian when $n > 1$. In particular, the higher dimensions allow us to define multiple operations, whereas π_1 does not have that sort of flexibility. In particular, we might notice how there are two natural ways to concatenate things with S^1—e.g., $\gamma \circ \gamma'$ might be doing γ first and then γ' second, or it might be doing γ second and γ' first, but this doesn't actually satisfy the interchange operation.

Well, the same argument from before about a topological group having two operations implying it is abelian still applies here! So, π_2 and higher are abelian groups. People saw this group and thought that this was crazy—how can they all be abelian? But, no, this is a correct and useful definition.

I mean, an alternative explanation is just to see that we can shift the domains around. That's a property of higher dimensions, which give us more flexibility. However, that idea isn't as rigorous, I guess.

So, we gain a group structure when going from π_0 to π_1. And, from π_2 onward, we gain commutativity! It seems that in higher dimensions, there is a ton more structure and that is really interesting.

As some random examples, it turns out $\pi_2(S^2) \cong \mathbb{Z}$ by the same helix argument as before. Indeed, $\pi_n(S^n) \cong \mathbb{Z}$ for any positive n. Meanwhile, $\pi_1(S^2) \cong 0$ as we saw earlier. So, the next most interesting question is $\pi_3(S^2)$?

This is $\mathbb{Z}$, but that is super hard to motivate it. In fact, $\pi_3(S^2)$ being nontrivial was an influential early example that made the homotopy groups of spheres something interesting to study. Further, it is an example of what's called a fiber bundle. Well, how can we get some intuition for why $\pi_3(S^2) \cong \mathbb{Z}$ as well?

Well, what does it mean for $\pi_3(S^2)$ to be non-trivial? It means that we can map S^3 (a three-dimensional hypersphere embedded in four-dimensional space) to S^2 (a regular sphere) in a non-trivial way, i.e., something so complicated that it is not homotopic to the trivial map.

Before we begin to understand that map, let's talk about what properties it has. First, for every point on S^2, a whole great circle of S^3 maps to it. That's pretty cool, but it also gives us the term "fiber bundle."

Basically, our idea is as follows. The space S^3 looks like a bunch of circles (S^1), one for each point on a sphere (S^2). These circles are called fibers, and this idea of associating one space to every point on another space is called a fiber bundle. This is analogous to how the Möbius strip looks like a line segment attached to each point of a circle (the circle is called a base), except there are global properties like a twist that are not captured locally. In both cases, the topological space is locally a product of some base and fiber, but there are subtle global properties. Fiber bundles help us study such constructions systematically.

That is super cool, but we can try going back to understanding the Hopf fibration.

Well, we start with a pair of complex numbers z_0 and z_1 where $|z_0|^2 + |z_1|^2 = 1$. That's the same as S^3 if we expand it out, because we get $a^2 + b^2 + c^2 + d^2 = 1$. Specifically, this creates a three-dimensional surface because we have four variables and one equation.

In other words, $\mathbb{R}^4$ can be thought of as $\mathbb{C}^2$, while S^3 is a subset of $\mathbb{C}^2$. Similarly, we can write $\mathbb{R}^3$ as $\mathbb{C} \times \mathbb{R}$.

So, how might we sort of map $\mathbb{C}^2$ downward onto $\mathbb{C} \times \mathbb{R}$ so that the result still has unit norm? It turns out that $(z_0, z_1) \mapsto (2z_0\overline{z_1}, |z_0|^2 - |z_1|^2)$ will give us what we want, and it may easily be seen that this preserves norm. Next, it's clear that this map is continuous. In order to verify that this map is not homotopic to the trivial loop, we need to use some scary invariants, but that's fine.

We also have the nice fibration property that the preimage of every point on S^2—the set of things that map to any given point on the sphere—is a circle on S^3. In fact, it's a great circle, and the algebra shows that.

The Hopf fibration turns out to be a generator for $\pi_3(S^2)$. Moreover, it has order ∞, so that's how we show that the group is both infinite and cyclic—$\mathbb{Z}$.

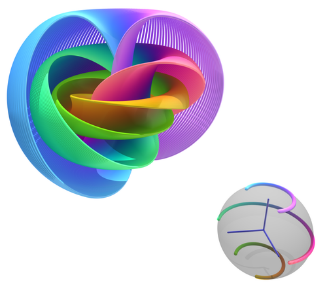


Figure 66: Niles Johnson Hopf Fibration CC BY 4.0

And, the Hopf fibration showed something deeper about the distinction homology and homotopy groups. Early topologists were interested in studying holes just like us, and their first major tool was the fundamental group. However, the Hopf fibration suggests that homotopy captures topological features more subtle than just holes, which prompted further study of homology theory.

What about the general case? This was just for $\pi_3(S^2)$.

Let's look at a table called the homotopy group of spheres. **It's really scary.**

Well, computing those homotopy groups is actually really hard. Perhaps it will help to find some intuition between π_n and π_m. In fact, let's just start with π_n and π_{n+1}.

But, to motivate that, we can go as simple as π_1 and π_2. How do we visualize maps from π_2 to some space $\underline{\overline{X}}$?

In particular, is there maybe some way to construct loops in π_2 in terms of π_1? Maybe if we build up the dimensions, we can understand higher homotopy groups in terms of lower ones. Can maps from S^2 be understood in terms of maps in S^1?

Perhaps we'll notice that S^2 is sort of like a loop of loops.

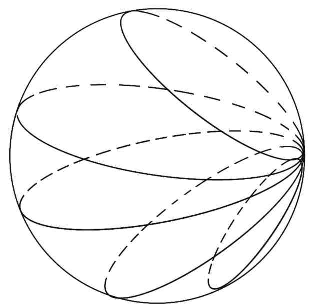


Figure 67: S^2 is a loop of loops

In particular, $\pi_2(\underline{\overline{X}}) \cong \pi_1$(all the loops on $\underline{\overline{X}}$). That really does give us a loop of loops because the loops loop back on each other—if we start at the trivial loop and move around the sphere, we'll get back to the trivial loop at the end.

Let's explore this further. What really is the space of all loops on $\underline{\overline{X}}$?

Hang on, isn't that just $\pi_1(\underline{\overline{X}})$? Well, not quite because π_1 considers the loops up to homotopy. So, π_1 would not be the right space. Instead, we need to define some new notation.

Loop Space

The **loop space** $\Omega(\underline{\overline{X}})$ is the set of all loops.

Now, it does have a bit more structure than that. Specifically, it is a topological space under what's called the compact-open topology. Loosely, two functions are close if they don't vary very much (a sketch of the rigorous definition is that the open sets are generated by the functions who, at least on some compact set, belong to the same open neighborhood), but we won't get any deeper into that.

Well, how can we relate Ω to π_1? They do seem to be not too far apart. (By the way, you will sometimes see $\Omega\underline{\overline{X}}$ instead of $\Omega(\underline{\overline{X}})$ since mathematicians get lazy.)

Timothy

As bad as Aareyan is at drawing curly braces, topologists are worse at drawing parentheses.

Well, we could say that $\pi_1(\underline{\overline{X}})$ is just $\Omega(\underline{\overline{X}})$ modulo the equivalence relation of homotopy. Even fancier, we might write $\pi_1(\underline{\overline{X}}) = \pi_0(\underline{\overline{X}})$, where two loops are connected if there is a homotopy between them.

Well anyways, it turns out that looping, i.e., constructing the loop space, sort of lowers the dimension. Even though it seems a lot more complicated, the act of looping lowers each homotopy group by a slot, so it is almost as if the dimension as a whole is lowered.

Wait, if SUS raises the dimension and looping does the opposite, then is it possible that they combine to do something special? It turns out that that looping and suspending are roughly complementary, but we'll see a more formal statement of that later.

By the way, another perspective for this is de-looping. Basically, it's the real inverse of looping, so if $\underline{\overline{Y}}$ is homotopic to $\Omega(\underline{\overline{X}})$, then we can say that $\underline{\overline{X}} = B\underline{\overline{Y}}$. Then, $\underline{\overline{X}}$ would be the de-looping of $\underline{\overline{Y}}$. This essentially lowers homotopy groups, and we'll see that more later.

So now, we know how to describe π_2 in terms of something that is sort of like π_1. What about π_{n+1} in terms of π_n?

It's the same! We can think of S^{n+1} as a loop of S^n's, which tells us that $\pi_{n+1}(\underline{X}) \cong \pi_n(\Omega(\underline{X}))$.

These relations are super cool! And, we use the fact that S^{n+1} and S^n are related. Specifically, that relation is SUSpension, but we can define SUS for any topological space.

So, we can start by generalizing the homotopy groups. In particular, we can have any map to $\underline{X}$ and use a shape different from S^n.

Homotopy Classes of Maps

For two topological spaces $\underline{Z}$ and $\underline{X}$, the **homotopy class** $[\underline{Z}, \underline{X}]$ is the set of maps from $\underline{Z} \to \underline{X}$ up to homotopy.

Is this a group? Remember that a group has an operation $m\colon G \times G \to G$, an inverse 🐸$^{-1}\colon G \to G$, and an identity $e : * \to G$, which may be unfamiliar notation and not everything, but we like to define things in terms of these maps.

Well, this is not a group in general, but S^1 is both a group and a cogroup. Whenever you see *co-*, that just means to reverse all the arrows. In particular, the inverse map is the same, and e is also easy, but m has comultiplication (where we go from G to something called $G \vee G$). You need to check coassociativity, coinverses, and counits. That's scary.

But anyways, this is where it goes back to the SUSpension. We know that $\mathrm{SUS}(S^n)$ is just $\mathrm{SUS}(S^{n+1})$, but the more general case is that $[\mathrm{SUS}(Z), X] \cong [Z, \Omega(X)]$.

This relation is called adjunction, specifically the loop-SUSpension adjunction. This is sort of a fundamental relation, and relates to something called the Eckmann-Hilton duality, but it is not even that hard to prove in analogy with what we had before!

In fact, exploiting this result, we can show a very nice series of results. First, $\pi_n(S^m) \cong \pi_{n+1}(S^{n+1})$ when $m > n/2$, which is super cool (and that's a consequence of the Freudenthal suspension theorem). Another way to state this is that $\pi_{n+k}(S^n)$ is independent of n when $n \geq k+2$, and that is called stabilization, which is a general principle in algebraic topology.

Stable homotopy groups are crucial in understanding the general case and suggest that the structure of all homotopy groups are very intricate.

This gets into things like spectra, which are sequences of spaces that satisfy some really cool properties. Things are wild in lower dimensions, but we tend to find shelter in higher dimensions. Somehow, if we keep applying SUS, we get rid of a lot of noise information and are just left with some fundamental infrogmation.

Here's something scary that topologists talk about. A spectrum $\mathbb{S}$ is an infinite collection of topological spaces that only has stable behavior and no unstable behavior. That is, $\mathbb{S}$ satisfies

$$\pi_n(\mathbb{S}) = \lim_{k\to\infty} \left(\prod_{n+k} S^k \right).$$

This is a limit not in terms of analysis, but just like a formal definition. But anyway, that's the sphere spectrum. The topologists say that the integers are a mistake, and we should work with the sphere spectrum instead.

But, at least it is somewhat helpful. For instance, we earlier said that SUSpension and looping are inverses in some sense. Well, if we work in spectra, then they really are inverses. In particular, let's look at the Eilenberg-MacLane spectrum. We can define $K(G, n)$ as some sort of class of topological spaces, and these play really nicely with SUSpension and looping. Specifically, they sort of increment and decrement 1, respectively (under certain conditions). Further, this leads to the Brown representation theorem.

> **Anonymous**
>
> That sounds scary.

> **Aareyan**
>
> So, you hear "brown" and "representation" in the same sentence and you feel scared?

Brown Representation Theorem

It turns out that the cohomology group $H^n(\underline{\overline{X}}, G)$ is isomorphic to $[\underline{\overline{X}}, K(G, n)]$.

Remember, the superscript is for cohomologies. This is just one of the many places that cohomology pops up. Intuitively, we have $H^n(S^m, G)$, which is G if $n = m$ and 0 everywhere else. That leads to

$$[S^m, K(G, n)] = \pi_m(K(G, n)) = \begin{cases} G & m = n, \\ 0 & \text{else.} \end{cases}$$

> **Connor**
>
> This talk has been very successful in convincing people not to do topology.

The idea is that cohomology and homology are much more general than topology, but topology is just where it first comes up. Anyways, that's so cool! We've seen how SUSpension and looping pop up in very many places and relate to more abstract classes.

Anyways, that concludes *Math Until We Die*. Stick around for the photo!