There is a non-Connes embeddable Equivalence Relation

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August 5, 2025

What is Connes Embedding I

Definition

A group Γ is **residually finite dimensional (RFD)** if for each $g \neq 1 \in \Gamma$ there exists a $*$-homomorphism $\phi_g : \Gamma \to U_{n_m}$ with $\phi_g(g) \neq 1$.

- Equivalently, there is an embedding $G \hookrightarrow \prod_n U_n$.
- Let $\Gamma = \langle a, b, c, d | b^a = b^2, c^b = c^2, d^c = d^2, a^d = a^2 \rangle$. All finite dimensional reps of Γ are trivial.

What is Connes Embedding II

- Replace $*$-homomorphism with **approximate $*$-homomorphisms**
- an approximate $*$-homomorphism is a sequence $\phi_n : \Gamma \to U_{k_n}$ with

$$\left\|\phi_n(gg') - \phi_n(g)\phi_n(g')\right\|_2 \to 0 \forall g, g' \in \Gamma.$$

- If approximate representations seperate points of G it is **Connes embeddable**
- This is a property of group von Neumann algebra $L(G)$, and generalizes to tracial von Neumann algebras.

What is Connes Embedding III

- equivalently $M \hookrightarrow \prod_U M_n(\mathbb{C})$ or $M \hookrightarrow R^U$ trace preservingly.
- Can lift **first order universal sentences** from finite dimensional algebras.

Theorem (Ji-Natarajan-Vidick-Wright-Yuen'20)

There is a II_1 factor that is not Connes embeddable.

- Goal: Narrow down this II_1 factor.
- Are all $L(\Gamma)$ Connes embeddable?

Theorem (M'25)

There is a Property (T) p.m.p ergodic equivalence relation $\mathcal{R}$ so that $L(\mathcal{R})$ is not Connes embeddable.

Reduction to MIP*=RE I

- Let M be a seperable tracial von Neumann algebra.
- There is a trace τ on $\mathcal{A} = C^*(\mathbb{F}_\infty)$ with $\tau(\mathcal{A})'' = M$.
- M is connes embeddable precisely when τ is an **amenable trace**.
- for $\mathbb{F}_\infty$ this means $\tau = \lim \tau_n$ where τ_n have finite dimensional GNS.
- CEP reduces to if $T(\mathbb{F}_\infty) = AT(\mathbb{F}_\infty)$.

Reduction to MIP*=RE II

- Let $A \subset B$ be convex sets.
- let I be a countable index set.
- $T(\mathfrak{G}) : B \to [0,1]$ for $\mathfrak{G} \in I$ be affine maps.
- Let $\omega^{co}(\mathfrak{G}) = \sup_B T(\mathfrak{G}, b)$ and $\omega^*(\mathfrak{G}) = \sup_A T(\mathfrak{G}, a)$.
- There is a computably enumerable sequence $\omega^*_d(\mathfrak{G}) \nearrow \omega^*(\mathfrak{G})$.
- There is a computable enumerable sequence $\omega^{co}_d(\mathfrak{G}) \searrow \omega^{co}(\mathfrak{G})$
- Suppose $\omega^*(\mathfrak{G})$ is uncomputable.
- Then $A \neq B$.

Reduction to MIP*=RE III

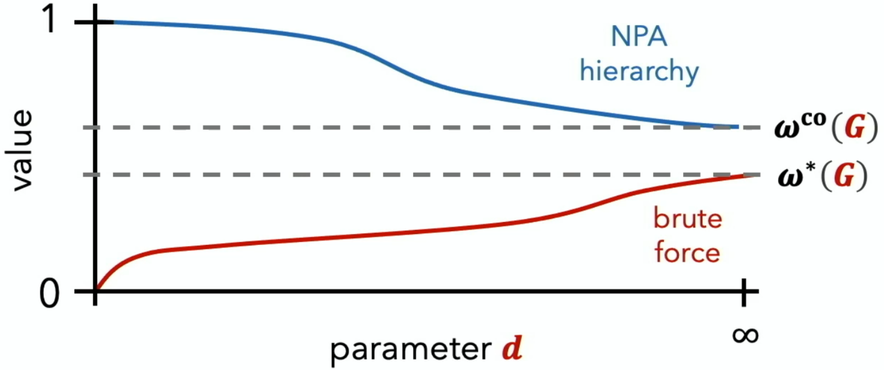


Figure: credit: John Wright

Invariant random subgroups

- Let Γ be a discrete countable group.
- Let $\mathrm{sub}(\Gamma) \subset \{0,1\}^{\Gamma}$ be the set of subgroups of Γ.
- A **Random Subgroup** is a Borel measure $H \in \mathrm{Prob}(\mathrm{sub}(\Gamma))$
- An **Invariant Random Subgroup(IRS)** is a random subgroup invariant under conjugation action of Γ on $\mathrm{sub}(\Gamma)$.

Examples of IRS

1. Let $N \triangleleft \Gamma$ be normal subgroup, then δ_N is an IRS.
2. Let α be a p.m.p action of Γ on (X, μ).
   - Define $\nu_\alpha \in \mathrm{IRS}(\Gamma)$ as

     $$\nu_\alpha(A) = \mu(x \in X : \mathrm{Stab}(x) \in A)$$

     for $A \subset \mathrm{sub}(\Gamma)$.
   - Pick $x \in X$ randomly and look at $\mathrm{Stab}(x)$.
   - If we consider $\mathrm{Stab} : X \to \mathrm{sub}(\Gamma)$ then this is $\mathrm{Stab}_*(\mu)$.

Examples of IRS II

Theorem

All IRS of countable groups arise as above.

3. If $H \in \mathrm{IRS}(\Gamma)$ arises from an action on a finite set, then we call it **finitely described**.
4. If $H \in \mathrm{IRS}(\Gamma)$ is the weak* limit of finitely described IRS then we call it **co-sofic**.

proposition

Let $N \lhd \Gamma$ where Γ is a free group. Then δ_N is co-sofic $\iff$ Γ/N is sofic.

Aldous-Lyons

- Sofic groups are groups whose approximate homomorphisms into $S_n \subset U_n$ seperate points.

Theorem (Bowen-Chapman-Lubotzky-Vidick '24)

There is a non co-sofic IRS on any free group.

- $\mathrm{IRS}(\Gamma) \neq \mathrm{IRS}_{sof}(\Gamma)$
- They did this by reducing this first to subgroup tests and then to MIP$*$=RE, which was cumbersome and took 50 pages.
- Using operator algebraic method, I can directly do this in about 5 pages.

Quotienting by an IRS

- For $H \in \mathrm{IRS}(\Gamma)$ define a trace $\tau_H(g) := \mathbb{P}(g \in H)$.
- If H arises from action of Γ on (X, μ) then $\tau_H(g) = \mu(gx = x)$.

proposition

Let $H = \delta_N$, then $\tau_H(\Gamma)'' = L(\Gamma/N)$.

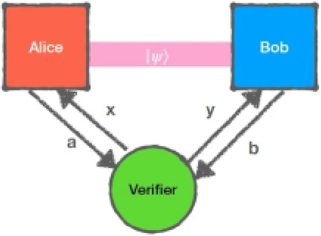
Definition

For $H \in \mathrm{IRS}(\Gamma)$, define $L(\Gamma/H) := \tau_H(\Gamma)''$.

proposition

If H comes from $\alpha : \Gamma \curvearrowright (X, \mu)$ then $L(\Gamma/H) \subset L(\mathcal{R}_\alpha)$.

Non-Local Games



- Finite question set Q and answer set A.
- Verifier samples question pair (x, y) from $Q \times Q$ according to π.
- Alice and bob answer $(a, b) \in A \times A$.
- Verifier accepts if $V(a, b|x, y) = 1$.

Strategies

- A strategy or corelation is the conditional distribution $(p(a, b|x, y))_{a,b,x,y}$.
- Strategies come from states on $C^*(\mathbb{F}_{Q,A})$ with

$$p(a, b|x, y) = \tau(e_x^a e_y^b).$$

- If the states are traces, these are **quantum commuting strategies**.
- If the states are f.d. traces these are **quantum** strategies. If the states are amenable traces then these are **quantum approximate**.

IRS strategies

Definition

An **IRS strategy** on $\mathfrak{G}$ is a strategy coming from a trace τ on $C^*(\mathbb{F}_{Q,A}) \otimes C^*(\mathbb{Z}/2\mathbb{Z})$ with

$$p(a, b|x, y) = \tau((1 - J)e_x^a e_y^b)$$

A **permutation strategy** is an IRS strategy coming from a f.d. IRS.

Theorem (M'25)

There is a game $\mathfrak{G}$ with $\omega_{IRS}(\mathfrak{G}) > \omega^*(\mathfrak{G})$

Corollary

There is a non-amenable trace τ on a free group coming from an IRS.

The computability result

Theorem (Bowen, Chapman, Vidick '24)

For each Turing machine M there is computable map $M \mapsto \mathfrak{G}_M$ such that:

- *If M halts than $\mathfrak{G}_M$ has a perfect permutation strategy.*
- *If M does not halt then $\omega^*(\mathfrak{G}_M) < 1/2$.*