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Ultraproducts of C*-algebras

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Dedicated to the memory of Professor Béla Szőkefalvi-Nagy,
a great mathematician and a great human being

We initiate a general study of ultraproducts of C*-algebras, including topics on representations, homomorphisms, isomorphisms, positive linear maps and their ultraproducts. We partially settle a question of D. McDuff by proving, for a finite von Neumann algebra with a separable predual, that the continuum hypothesis implies the isomorphism of all of its tracial ultrapowers with respect to different free ultrafilters on the natural numbers. The analog for C*-ultrapowers of separable C*-algebras is equivalent to the continuum hypothesis. We also prove a finite local reflexivity theorem for operator spaces that implies that the second dual of a C*-algebra can be embedded in some ultrapower of the algebra using a completely positive completely isometric map.

1. Introduction

Ultraproducts were first introduced in model theory and played a fundamental role in understanding elementary equivalence and constructing nonstandard models. They were then applied to many areas of mathematics. For example, H. G. Dales and W. H. Woodin [DW] used ultraproducts to prove the independence from the axioms of set theory (ZFC) of the assertion that every algebra homomorphism from a commutative C*-algebra into a Banach algebra is continuous. A version of ultraproducts for Banach spaces was defined and used to construct important examples and settle a number of important questions (see [He]). The ultrapower construction for finite von Neumann algebras was introduced by D. McDuff [Mc]

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and G. Janssen [J] independently. McDuff proved that if the relative commutant of a factor of type II_1 in its ultrapower is noncommutative, then the factor is isomorphic to the tensor product of the factor with the hyperfinite II_1 factor. Elements in the relative commutant of an algebra in its ultrapower correspond to central sequences in the algebra. Central sequences and ultraproducts played a very important role in Connes's work on the classification of injective von Neumann algebras and their automorphisms (see, e.g., [C]). Some aspects of Connes's results were extended to C*-algebras by Phillips [Ph]. Akemann and Pedersen [AP] proved that a separable C*-algebra has continuous trace if and only if it has no nontrivial central sequences. Ultraproducts of C*-algebras are closely related to approximation properties of elements in the algebras. Several people have used ultraproducts of C*-algebras as technical tools to study problems in C*-algebras (see, e.g., [Ha]). In this paper, we initiate a general study of ultraproducts of C*-algebras.

In Section 2, we define a generalized notion of ultraproduct that contains the classical notions and those used in Banach spaces and von Neumann algebras as special cases. The generalized notion of ultraproducts has uses in K-theory, since the K-groups of a C*-ultraproduct are rarely the algebraic ultraproduct of the K-groups, but they can often be expressed as generalized ultraproducts. We also summarize some of the basic properties of ultraproducts of C*-algebras. Two isomorphism problems on ultrapowers are studied in Section 3. First, assuming the continuum hypothesis, we answer affirmatively the problem whether ultrapowers of a separable C*-algebra (or a finite von Neumann algebra with a separable predual) on all free ultrafilters on $\mathbb{N}$ are all $*$-isomorphic. Then we study the problem of C*-algebras with isomorphic ultrapowers, which is related to embeddings of C*-algebras (or finite von Neumann algebras) into certain ultrapowers. In Section 4, we prove a version of local reflexivity for operator spaces and extend a classical result for Banach spaces by showing that the second dual $\mathfrak{A}^{\#\#}$ of a C*-algebra $\mathfrak{A}$ can be isometrically embedded into an ultrapower of $\mathfrak{A}$ where the embedding is completely positive (but not usually a $*$-homomorphism) such that the range of the embedding is the range of a completely positive idempotent on the ultrapower. Exact sequences of C*-algebras are preserved by taking ultraproducts. We also show that nontrivial ultraproducts are usually highly nonseparable. In Section 5, spatial ultraproducts of C*-algebras, ultraproducts of states and representations, and simplicity of ultraproducts are discussed. We prove a generalization of Kadison's transitivity theorem for certain ultraproducts with finite-dimensional spaces replaced by separable ones. We show that an ultraproduct of irreducible representations (respectively, pure states, primitive C*-algebras) is irreducible (respectively, pure, primitive), and that ultraproducts of simple C*-algebras need not be simple, but in certain cases they are. The last section contains discussions on ultraproducts with respect to ultrafilters closed under countable intersections. Strange things happen. This leads to an ultraproduct characterization of the existence of measurable cardinals.

2. Definitions and basic properties

Suppose $\mathbb{I}$ is an infinite set. An *ultrafilter* α on $\mathbb{I}$ is a collection of subsets of $\mathbb{I}$ such that the empty set $\emptyset \notin \alpha$, α is closed under finite intersections, and, for each subset A of $\mathbb{I}$, either $A \in \alpha$ or $\mathbb{I} \setminus A \in \alpha$. One example of an ultrafilter is obtained by choosing an ι in $\mathbb{I}$ and letting α be the collection of all subsets of $\mathbb{I}$ that contain ι. Such an ultrafilter is called a *principal* ultrafilter; ultrafilters not of this form are called *free.*

In general, there are two different types of free ultrafilters we must consider. We will call a free ultrafilter α *countably cofinal* if there is a sequence $\{A_n\}$ in α whose intersection is empty. Otherwise, α is called $\aleph_1$*-complete* [CN]. Free ultrafilters on a countable set are always countably cofinal. The existence of free $\aleph_1$-complete ultrafilters is an open problem (equivalent to the existence of measurable cardinals [CN]). We will call an ultrafilter *nontrivial* if it is both free and countably cofinal. We leave some of the discussions on ultraproducts with respect to $\aleph_1$-complete ultrafilters to the last section (Section 6).

Nontrivial ultrafilters on a countable set $\mathbb{I}$ can be identified as points in $\beta(\mathbb{I}) \setminus \mathbb{I}$, where $\beta(\mathbb{I})$ is the Stone–Céch compactification of $\mathbb{I}$ (when $\mathbb{I}$ is endowed with the discrete topology). One may also view $\beta(\mathbb{I})$ as the compact Hausdorff space so that $l^\infty(\mathbb{I})$ (uniformly bounded complex-valued functions on $\mathbb{I}$) is $*$-isomorphic to $C(\beta(\mathbb{I}))$ as C*-algebras. One important property of $\beta(\mathbb{I})$ is that every bounded function on $\mathbb{I}$ extends to a continuous function on $\beta(\mathbb{I})$.

Suppose $\mathcal{X}$ is another set, $f\colon \mathbb{I} \to \mathcal{X}$ is a mapping and $E \subseteq \mathcal{X}$. We say that $f(\iota)$ is *eventually in* E along α if $f^{-1}(E) = \{\iota \in \mathbb{I} : f(\iota) \in E\} \in \alpha$. If $\mathcal{X}$ is a topological space, we say that $f(\iota)$ converges to x (in $\mathcal{X}$) along α, denoted by $\lim_{\iota\to\alpha} f(\iota) = x$, if $f(\iota)$ is eventually in each neighborhood of x. It is a well-known fact about ultrafilters that if $\mathcal{X}$ is a compact Hausdorff space, then $\lim_{\iota\to\alpha} f(\iota)$ always exists in $\mathcal{X}$ for every $f\colon \mathbb{I} \to \mathcal{X}$ and every ultrafilter α on $\mathbb{I}$.

Now we review, briefly, some of the existing constructions of ultraproducts of sets, groups, algebras, Banach spaces, C*-algebras, etc.

First, let $\mathcal{Y}_\iota$, $\iota \in \mathbb{I}$, be sets, $\prod_{\iota\in\mathbb{I}} \mathcal{Y}_\iota$ or $\prod_\iota \mathcal{Y}_\iota$ be the Cartesian product of $\mathcal{Y}_\iota$'s. Elements in $\prod_{\iota\in\mathbb{I}} \mathcal{Y}_\iota$ are denoted by $\{Y_\iota\}$ for Y_ι in $\mathcal{Y}_\iota$. Let α be a nontrivial ultrafilter on $\mathbb{I}$. We define an equivalence relation $\sim$ on $\prod_{\iota\in\mathbb{I}} \mathcal{Y}_\iota$ by $\{Y_\iota\} \sim \{Z_\iota\}$ if and only if $Y_\iota = Z_\iota$ eventually along α. Let $\prod^\alpha \mathcal{Y}_\iota$ be the equivalence classes of elements $\{Y_\iota\}$ in $\prod_{\iota\in\mathbb{I}}\mathcal{Y}_\iota$ with respect to $\sim$ and α. In this case, $\prod^\alpha\mathcal{Y}_\iota$ is the original ultraproduct defined in model theory [CK], which we call the *classical ultraproduct* of the $\mathcal{Y}_\iota$'s. The equivalence class of an element $\{Y_\iota\}$ in $\prod_\iota\mathcal{Y}_\iota$ (with respect to $\sim$ and α) is denoted by $[\{Y_\iota\}]$ (or $\{Y_\iota\}_\alpha$). Note that if the $\mathcal{Y}_\iota$'s have certain additional structure, e.g., they are groups or partially ordered sets, then $\prod^\alpha\mathcal{Y}_\iota$ inherits this structure in the obvious way, i.e., the Cartesian product $\prod_\iota\mathcal{Y}_\iota$ has the same structure and $\sim$ is a congruence on $\prod_\iota\mathcal{Y}_\iota$.

Next, we assume that each $\mathcal{Y}_\iota$ is a Banach space. In this case, the ultraproduct $\prod^\alpha\mathcal{Y}_\iota$ of the $\mathcal{Y}_\iota$'s as Banach spaces is the l^∞-product of the $\mathcal{Y}_\iota$'s modulo the closed subspace $\sum^\alpha\mathcal{Y}_\iota$ of all elements $\{\xi_\iota\}$ that converge in norm to 0 along α. This ultraproduct is the ultraproduct of Banach spaces discussed in [He]. Note that $\|[\{u_\iota\}]\| = \lim_{\iota\to\alpha}\|u_\iota\|$ defines a norm on $\prod^\alpha\mathcal{Y}_\iota$. If each $\mathcal{Y}_\iota$ is a Banach algebra or a C*-algebra, then so is $\prod^\alpha\mathcal{Y}_\iota$. More interestingly, if each $\mathcal{Y}_\iota$ is a Hilbert space, then so is $\prod^\alpha\mathcal{Y}_\iota$ with the inner product $\langle[\{u_\iota\}],[\{v_\iota\}]\rangle = \lim_{\iota\to\alpha}\langle u_\iota, v_\iota\rangle$.

Now, suppose that each $\mathcal{Y}_\iota$ is a finite von Neumann algebra with a faithful normal tracial state τ_ι. Let $\prod_\iota\mathcal{Y}_\iota$ be the l^∞-product of the $\mathcal{Y}_\iota$'s. Then $\prod_\iota\mathcal{Y}_\iota$ is a von Neumann algebra (with pointwise multiplication). If $\{X_\iota\}$ and $\{Y_\iota\}$ are two elements in $\prod_\iota\mathcal{Y}_\iota$, then we define $\{X_\iota\}\sim\{Y_\iota\}$ when $\lim_{\iota\to\alpha}\|X_\iota - Y_\iota\|_2 = 0$, where $\|X\|_2 = \tau_\iota(X^*X)^{1/2}$ is the trace norm given by τ_ι. Then $\prod^\alpha\mathcal{Y}_\iota$ is $\prod_\iota\mathcal{Y}_\iota$ modulo the equivalence relation $\sim$. We will call such ultraproducts *tracial ultraproducts*.

When $\mathcal{Y}_\iota = \mathcal{Y}$ for all ι, then $\prod^\alpha\mathcal{Y}_\iota$ is called the *ultrapower* of $\mathcal{Y}$, denoted by $\mathcal{Y}^\alpha$. For example, McDuff [Mc] considered certain tracial ultrapowers of factors of type II_1 along free ultrafilters on $\mathbb{N}$. A tracial ultrapower of a finite von Neumann algebra with respect to a faithful normal trace is again a von Neumann algebra [Sa].

In the following, we introduce a general construction of ultraproducts. In order to avoid getting bogged down in logic and model theory, we consider the ultraproduct construction at the level of sets and elementary algebraic structures. For the reader interested in the logical aspect of ultraproducts, we refer the reader to [CK] and [CN].

Suppose, for each ι in $\mathbb{I}$, $\mathcal{Y}_\iota$ is a nonempty set. Suppose $\mathcal{E}$ is a collection of mappings on $\mathbb{I}$, such that, for each ι in $\mathbb{I}$, and each E in $\mathcal{E}$, $E(\iota)$ defines a nonnegative function on $\mathcal{Y}_\iota$, i.e., $E(\iota)\colon\mathcal{Y}_\iota\to[0,\infty]$. Also suppose $\mathcal{F}$ is a family of mappings on $\mathbb{I}$ such that, for each ι in $\mathbb{I}$, and each F in $\mathcal{F}$, $F(\iota)\colon\mathcal{Y}_\iota\times\mathcal{Y}_\iota\to[0,\infty]$ is a semi-metric on $\mathcal{Y}_\iota$. We define the $\mathcal{E}$-bounded Cartesian product of $\{\mathcal{Y}_\iota : \iota\in\mathbb{I}\}$, denoted by $\prod_{\mathcal{E}}\mathcal{Y}_\iota$, to be the set of $Y = \{Y_\iota\}_{\iota\in\mathbb{I}}$ in the Cartesian product of the $\mathcal{Y}_\iota$'s such that, for each E in $\mathcal{E}$, $\{E(\iota)(Y_\iota)\}$ is bounded on $\mathbb{I}$. Let α be an ultrafilter on $\mathbb{I}$. We define an equivalence relation $\sim_{\mathcal{F}(\alpha)}$ on $\prod_{\mathcal{E}}\mathcal{Y}_\iota$: $\{Y_\iota\}\sim_{\mathcal{F}(\alpha)}\{Z_\iota\}$ if and only if, for each F in $\mathcal{F}$, $\lim_{\iota\to\alpha}F(\iota)(Y_\iota, Z_\iota) = 0$. We define the *ultraproduct* of the $\mathcal{Y}_\iota$'s along α with respect to $(\mathcal{E}, \mathcal{F})$, denoted by $\prod_{\mathcal{E},\mathcal{F}}^{\alpha} \mathcal{Y}_\iota$, to be $(\prod_{\mathcal{E}} \mathcal{Y}_\iota)/\sim_{\mathcal{F}(\alpha)}$. If we have that each $\mathcal{Y}_\iota = \mathcal{Y}$, we call the ultraproduct $\prod_{\mathcal{E},\mathcal{F}}^{\alpha} \mathcal{Y}_\iota$ an *ultrapower* of $\mathcal{Y}$. This general notion of ultraproduct contains the previously mentioned examples.

For example, assume each $\mathcal{Y}_\iota$ is a Banach space, and suppose $\mathcal{E} = \{E\}$ and $\mathcal{F} = \{F\}$, such that, for each ι in $\mathbb{I}$ and each u, v in $\mathcal{Y}_\iota$, $E(\iota)(u) = \|u\|$ and $F(\iota)(u,v) = \|u - v\|$. In this case, $\prod_{\mathcal{E},\mathcal{F}}^{\alpha} \mathcal{Y}_\iota$ is the l^∞-product of the $\mathcal{Y}_\iota$'s modulo the closed subspace of all elements $\{\xi_\iota\}$ that converge in norm to 0 along α, thus it agrees with the above ultraproduct $\prod^{\alpha} \mathcal{Y}_\iota$ of $\mathcal{Y}_\iota$'s as Banach spaces.

Sometimes, it is useful to have this general notion of ultraproducts. It helps to describe objects that cannot be expressed by classical ultraproducts. For example, suppose each $\mathcal{Y}_\iota$ is an abelian group, $\mathcal{E} = \{E\}$ and $\mathcal{F} = \{F\}$, where $E(\iota)$ is the constant 1 function on $\mathcal{Y}_\iota$, and $F(\iota)$ is the discrete metric on $\mathcal{Y}_\iota$ (i.e., $F(\iota)(X,Y) = 1$ if $X \neq Y$ and $F(\iota)(X,Y) = 0$ if $X = Y$). Then $\prod_{\mathcal{E},\mathcal{F}}^{\alpha} \mathcal{Y}_\iota$ is the classical ultraproduct of $\mathcal{Y}_\iota$'s. If $\mathcal{E}_1 = \mathcal{E} \cup \{H\}$, where, for each ι in $\mathbb{I}$ and each Y in $\mathcal{Y}_\iota$, $H(\iota)(Y)$ is the order of Y in the group $\mathcal{Y}_\iota$, then $\prod_{\mathcal{E}_1,\mathcal{F}}^{\alpha} \mathcal{Y}_\iota$ is the torsion subgroup of $\prod_{\mathcal{E},\mathcal{F}}^{\alpha} \mathcal{Y}_\iota$.

Another more pertinent example comes from C*-algebraic K-theory. For each positive integer n, it is well-known that $K_0(M_n) = \mathbb{Z}$, where M_n is the C*-algebra of all $n \times n$ matrices. It is not difficult to show, for any nontrivial ultrafilter α on $\mathbb{N}$, that $K_0(\prod^{\alpha} M_n)$ is not the algebraic ultraproduct $\prod^{\alpha} \mathbb{Z}$. However, it is the generalized ultraproduct $\prod_{\mathcal{E},\mathcal{F}}^{\alpha} \mathbb{Z}$, where $\mathcal{E} = \{E\}$, $\mathcal{F} = \{F\}$ with $E(n)(z) = |z/n|$ and $F(n)$ is the discrete metric on $\mathbb{Z}$. We will discuss the K-theory of ultraproducts more fully in a later paper.

We end this section by summarizing some of the well-known, and useful facts concerning elements in ultraproducts of C*-algebras and their representatives.

PROPOSITION 2.1. *Let $\mathfrak{A}_\iota$, $\iota \in \mathbb{I}$, be unital C*-algebras and α an ultrafilter on $\mathbb{I}$. Then*

1. *if A is a self-adjoint operator in $\prod^{\alpha} \mathfrak{A}_\iota$, then there are self-adjoint operators A_ι in $\mathfrak{A}_\iota$ such that $A = [\{A_\iota\}]$;*
2. *if P is a projection in $\prod^{\alpha} \mathfrak{A}_\iota$, then there are projections P_ι in $\mathfrak{A}_\iota$ such that $P = [\{P_\iota\}]$;*
3. *if U is a unitary operator in $\prod^{\alpha} \mathfrak{A}_\iota$, then there are unitary operators U_ι in $\mathfrak{A}_\iota$ such that $U = [\{U_\iota\}]$;*
4. *if $P = \{P_\iota\}_\alpha$, $Q = \{Q_\iota\}_\alpha$ are in $\prod^{\alpha} \mathfrak{A}_\iota$ and each P_ι, Q_ι are projections and if $V \in \prod^{\alpha} \mathfrak{A}_\iota$ is a partial isometry with $V^*V = P$ and $VV^* = Q$, then there are V_ι in $\mathfrak{A}_\iota$ such that, eventually along α, $V_\iota^* V_\iota = P_\iota$ and $V_\iota V_\iota^* = Q_\iota$;*
5. *if $P = \{P_\iota\}_\alpha \in \prod^{\alpha} \mathfrak{A}_\iota$ and each P_ι is a projection, and if Q is a projection in $\prod^{\alpha} \mathfrak{A}_\iota$ such that $Q \leq P$, then there are projections $Q_\iota \in \mathfrak{A}_\iota$ with $Q_\iota \leq P_\iota$, such that $Q = \{Q_\iota\}_\alpha$.*

Proof. 1. If $A = [\{A_\iota\}]$ and A is selfadjoint, then $A = [\{(A_\iota + A_\iota^*)/2\}]$.

2. Let $\{A_\iota\}$ be a representative for the projection P. From 1, we may assume that each A_ι is selfadjoint. From $P^2 = P$, we have that $\lim_{\iota \to \alpha} \|A_\iota^2 - A_\iota\| = 0$. Let f be a continuous nondecreasing function on $\mathbb{R}$ such that $f(t) = 0$ when $t \le \frac{1}{4}$ and $f(t) = 1$ when $t \ge \frac{3}{4}$. Then we know that when $\|A_\iota^2 - A_\iota\| \le \frac{1}{8}$, the spectrum of A_ι lies in $[-\frac{1}{4}, \frac{1}{4}] \cup [\frac{3}{4}, \frac{5}{4}]$ and so $f(A_\iota)$ is a projection. It is easy to see that $\lim_{\iota \to \alpha} \|A_\iota - f(A_\iota)\| = 0$. Replacing A_ι by I when $\|A_\iota^2 - A_\iota\| > \frac{1}{8}$ and by $f(A_\iota)$ when $\|A_\iota^2 - A_\iota\| \le \frac{1}{8}$, we have that $P = [\{A_\iota\}]$ and each A_ι is a projection.

3. Similar to (2). We leave it as an exercise.

4. First write $V = \{A_\iota\}_\alpha$. Since $V = QVP$, we can assume $A_\iota = Q_\iota V_\iota P_\iota$ for every ι. Let $g\colon \mathbb{R} \to \mathbb{R}$ be continuous with $g(t) = 0$ when $0 \le t \le 1/4$, and $g(t) = 1/\sqrt{t}$ when $3/4 \le t \le 5/4$, and for each ι, let $V_\iota = g(A_\iota A_\iota^*)A_\iota$. Since $V = g(VV^*)V$, it follows that $V = \{V_\iota\}_\alpha$. Note that we still have $V_\iota = Q_\iota V_\iota P_\iota$. However, as in the proof of (2), we have, eventually along α, that $V_\iota V_\iota^*$ is a projection, $V_\iota V_\iota^* \le Q_\iota$, and $\|V_\iota V_\iota^* - Q_\iota\| < 1$, which implies that $V_\iota V_\iota^* = Q_\iota$. Similarly, $V_\iota^* V_\iota$ is eventually a projection dominated by P_ι but close to P_ι, which implies that, eventually along α, $V_\iota^* V_\iota = P_\iota$.

5. Write $Q = \{B_\iota\}_\alpha$ with $B_\iota = B_\iota^*$. Since $Q = PQP$, we can assume that $B_\iota = P_\iota B_\iota P_\iota$ for each ι. If we apply the technique of the proof of (2) above, letting Q_ι be $f(P_\iota B_\iota P_\iota)$, we see that, eventually along α, Q_ι is a projection and $Q_\iota \le P_\iota$.

□

3. Isomorphisms of ultrapowers

In this section, we study isomorphism problems of ultrapowers. First, a natural question about an ultrapower of a C*-algebra is whether it depends on the choice of the nontrivial ultrafilter. In the case $\mathbb{I} = \mathbb{N}$, we have the following result of H. J. Keisler [CK; 4.3] concerning classical ultrapowers.

THEOREM 3.1. *Assume that the continuum hypothesis holds. Suppose $\mathcal{R}$ is a universal algebra with cardinality at most $2^{\aleph_0}$ and having finitely many operations. Suppose also $\mathcal{S}$ is a countable subset of $\mathcal{R}$. If α, β are nontrivial ultrafilters on $\mathbb{N}$, then there is an isomorphism $\pi\colon \mathcal{R}^\alpha \to \mathcal{R}^\beta$ such that $\pi|\mathcal{S}$ is the identity (i.e., for every $S \in \mathcal{S}$, $\pi(\{(S, S, \ldots)\}_\alpha) = \{(S, S, \ldots)\}_\beta$).*

We can use Keisler's theorem to prove analogs results for more general ultrapowers.

THEOREM 3.2. *Assume the continuum hypothesis. Suppose $\mathcal{M}$ is a finite von Neumann algebra with a faithful finite normal trace τ. If $\mathcal{M}$ is trace-norm separable, then $\mathcal{M}^\alpha$ and $\mathcal{M}^\beta$ are $*$-isomorphic von Neumann algebras for any free ultrafilters α and β over $\mathbb{N}$, where $\mathcal{M}^\alpha$ and $\mathcal{M}^\beta$ are tracial ultrapowers of $\mathcal{M}$. Moreover, the relative commutant of $\mathcal{M}$ in $\mathcal{M}^\alpha$ is $*$-isomorphic to that of $\mathcal{M}$ in $\mathcal{M}^\beta$.*

Proof. We let $\mathcal{R}$ be the universal algebra $\mathcal{M}$ with the operations $+, \cdot, *$, and we define predicates "$\|x\| < 1$", "$\|x\|_2 < 1$" and "$x = iy$". Choose a countable trace-norm dense subset $\mathcal{S}$ of $\mathcal{M}$.

It follows from Theorem 3.1 that there is an isomorphism $\pi\colon \mathcal{R}^\alpha \to \mathcal{R}^\beta$ that fixes the elements of $\mathcal{S}$ and preserves the predicates. In particular, π maps $\{X \in \mathcal{R}^\alpha : \|X\| < 1\}$ and $\{X \in \mathcal{R}^\alpha : \|X\|_2 < 1\}$ onto $\{X \in \mathcal{R}^\beta : \|X\| < 1\}$ and $\{X \in \mathcal{R}^\beta : \|X\|_2 < 1\}$, respectively.

Note that the predicate $\|[\{X_n\}]\| < 1$ holds in $\mathcal{R}^\alpha$ (respectively, $\mathcal{R}^\beta$) if and only if, eventually along α (respectively, β), $\|X_n\| < 1$. We define

$$\mathcal{R}_b^\alpha = \{\{X_n\}_\alpha \in \mathcal{R}^\alpha : \lim_{n\to\alpha} \|X_n\| < \infty\}.$$

We define $\mathcal{R}_b^\beta$ similarly. Note that $\mathcal{R}_b^\alpha$ is the additive subgroup of $\mathcal{R}^\alpha$ generated by $\{X \in \mathcal{R}^\alpha : \|X\| < 1\}$. It follows that $\pi(\mathcal{R}_b^\alpha) = \mathcal{R}_b^\beta$.

Now, let $\mathcal{R}_0^\alpha = \{\{X_n\}_\alpha \in \mathcal{R}^\alpha : \lim_{n\to\alpha} \|X_n\|_2 = 0\} \cap \mathcal{R}_b^\alpha$ and $\mathcal{R}_0^\beta = \{\{X_n\}_\beta \in \mathcal{R}^\beta : \lim_{n\to\beta} \|X_n\|_2 = 0\} \cap \mathcal{R}_b^\beta$. It is clear that $\mathcal{R}_0^\alpha$ is the unique maximal additive subgroup contained in $\{X \in \mathcal{R}^\alpha : \|X\|_2 < 1\} \cap \mathcal{R}_b^\alpha$. It follows that $\pi(\mathcal{R}_0^\alpha) = \mathcal{R}_0^\beta$. Hence π induces an isomorphism $\rho\colon \mathcal{R}_b^\alpha/\mathcal{R}_0^\alpha \to \mathcal{R}_b^\beta/\mathcal{R}_0^\beta$.

But $\mathcal{R}_b^\alpha/\mathcal{R}_0^\alpha = \mathcal{M}^\alpha$ and $\mathcal{R}_b^\beta/\mathcal{R}_0^\beta = \mathcal{M}^\beta$. It is clear that ρ is bijective, additive, multiplicative $*$-mapping. Moreover, $\rho(iX) = i\rho(X)$ since π preserves the predicate "$Y = iX$". Also an element X in $\mathcal{M}^\alpha$ satisfies $\|X\| < 1$ (or $\|X\|_2 < 1$) if and only if X has a pre-image W in $\mathcal{R}_b^\alpha$ such that $\|W\| < 1$ (or $\|W\|_2 < 1$, respectively). Hence, for every X in $\mathcal{M}^\alpha$, $\|X\| < 1$ (or $\|X\|_2 < 1$) if and only if $\|\rho(X)\| < 1$ (or $\|\rho(X)\|_2 < 1$, respectively), which implies that ρ is an isometry and preserves trace(-norm). Since every additive isometry on a Banach space is linear over $\mathbb{R}$, it follows that ρ is linear over $\mathbb{C}$. Therefore ρ is a trace preserving $*$-isomorphism.

Since π fixes the elements of $\mathcal{S}$, which implies ρ fixes the points in (a trace-norm dense subset of) $\mathcal{M}$.

□

REMARK 3.3. The above theorem answers a question of D. McDuff [Mc]. The ideas in the proof of Theorem 3.2 can be used to prove analogs for Banach space ultrapowers or C*-ultrapowers.

COROLLARY 3.4. *Assume the continuum hypothesis. If $\mathfrak{A}$ is a C*-algebra with cardinality $2^{\aleph_0}$, then $\mathfrak{A}^\alpha$ and $\mathfrak{A}^\beta$ are $*$-isomorphic. Moreover, if $\mathfrak{A}$ is separable, there is a $*$-isomorphism from $\mathfrak{A}^\alpha$ to $\mathfrak{A}^\beta$ that fixes the elements of $\mathfrak{A}$. Thus the relative commutants of $\mathfrak{A}$ in $\mathfrak{A}^\alpha$ and $\mathfrak{A}^\beta$ are $*$-isomorphic.*

If the continuum hypothesis fails, the conclusion for Keisler's theorem (Theorem 3.1) no longer holds. However, P. Eklof [E] has shown that all nontrivial ultrapowers over $\mathbb{N}$ of the additive group $(\mathbb{Z},+)$ of the integers are isomorphic. On the other hand, if we view $\mathbb{Z}$ as an ordered structure with the usual ordering, A. Dow [D] and S. Shelah [S2] have independently proved that nonisomorphic ultrapowers exist:

THEOREM 3.5. *Assume the continuum hypothesis fails. Suppose $(X,\leq)$ is a partially ordered set containing an infinite chain. Then there are nontrivial ultrapowers α, β on $\mathbb{N}$ such that the classical ultrapowers $(X,\leq)^\alpha$ and $(X,\leq)^\beta$ are not order-isomorphic.*

To prove analogs of the preceding theorem we need to consider ordered structures associated with C*-algebras that are preserved under ultraproducts. The usual ordering on a C*-algebra fails in this regard, since the ordered structure of a C*-ultraproduct is not the classical ultraproduct of the ordered structures of the C*-algebras. One natural choice is the ordered K_0-group, but this too, except in certain cases, does not behave nicely with respect to ultraproducts; also the K_0-group throws away some of the natural structure on the projections in the algebra. A simpler choice comes from the partial ordering on the projections in the algebra induced by Murry–von Neumann equivalence.

Suppose $\mathfrak{A}$ is a C*-algebra and $P,Q\in\mathfrak{A}$ are projections. We say P and Q are Murry–von Neumann equivalent if there is a V in $\mathfrak{A}$ with $VV^*=P$ and $V^*V=Q$. We will say that two projections in $\mathfrak{A}$ are *equivalent* if each is Murry–von Neumann equivalent to a subprojection of the other. We let $\mathcal{P}(\mathfrak{A})$ denote the set of equivalence classes of projections in $\mathfrak{A}$, and we define a partial ordering $\leq$ on $\mathcal{P}(\mathfrak{A})$: $[P]\leq[Q]$ if and only if P is equivalent to a subprojection of Q. The fact that the functor $\mathcal{P}$ from the category of C*-algebras to the category of partially ordered sets behaves nicely with respect to ultraproducts is contained in the following lemma. To avoid confusion, we use our alternate notation $\{Y_\iota\}_\alpha$ for the usual equivalence class of an element $\{Y_\iota\}$ in an ultraproduct.

LEMMA 3.6. *Suppose $\{\mathfrak{A}_\iota:\iota\in\mathbb{I}\}$ is a family of C*-algebras and α is a nontrivial ultrafilter on $\mathbb{I}$. Then $\mathcal{P}(\prod^\alpha\mathfrak{A}_\iota)$ is order-isomorphic to the classical ultraproduct $\prod^\alpha\mathcal{P}(\mathfrak{A}_\iota)$.*

Proof. This all follows from Proposition 2.1. Projections P and Q in $\prod^\alpha \mathfrak{A}_\iota$ can be written as $P = \{P_\iota\}_\alpha$ and $Q = \{Q_\iota\}_\alpha$, where each P_ι and Q_ι are projections. Then $[P] \leq [Q]$ if and only if, eventually along α, $[P_\iota] \leq [Q_\iota]$. Thus the map $\sigma: \mathcal{P}(\prod^\alpha \mathfrak{A}_\iota) \to \prod^\alpha \mathcal{P}(\mathfrak{A}_\iota)$ defined by $\sigma(\{P_\iota\}_\alpha) = \{[P_\iota]\}_\alpha$ is an order-isomorphism. □

EXAMPLES. 1. Suppose $\mathcal{K}$ is the C*-algebra of all compact operators on a separable infinite-dimensional Hilbert space. All the projections in $\mathcal{K}$ have finite rank, and two projections are equivalent if and only if they have the same rank. Moreover $[P] \leq [Q]$ if and only if $\operatorname{rank}(P) \leq \operatorname{rank}(Q)$. Hence $\mathcal{P}(\mathcal{K})$ is order-isomorphic to $\{0, 1, 2, \ldots\}$ with the usual ordering.

2. Suppose $\mathfrak{A}$ is a Glimm algebra with supernatural number m. Since two projections in $\mathfrak{A}$ are equivalent if and only if they have the same trace, $\mathcal{P}(\mathfrak{A})$ is order-isomorphic to $\{\frac{k}{n} : k, n \in \mathbb{Z}, 0 \leq k \leq n, n \neq 0, n|m\}$ with the usual ordering.

3. Suppose $\mathfrak{A}_\theta$ is an irrational rotation algebra, where θ is the associated irrational number. Then $\mathcal{P}(\mathfrak{A}_\theta)$ is order-isomorphic to $\{m + n\theta \in [0,1] : m, n \in \mathbb{Z}\}$.

4. $\mathcal{P}(\mathcal{B}(\mathcal{H}))$ is order-isomorphic to $\mathbb{N} \cup \{\infty\}$, with the usual ordering, where $\mathcal{H}$ is the separable infinite-dimensional Hilbert space. Note that $K_0(\mathcal{B}(\mathcal{H})) = \{0\}$.

5. If X is a compact Hausdorff space, then $\mathcal{P}(C(X))$ is order-isomorphic to $(\{A \subset X : A \text{ is clopen}\}, \subset)$.

6. If $\mathfrak{A}$ is a von Neumann algebra, $\mathcal{P}(\mathfrak{A})$ is the set of Murray–von Neumann equivalence classes of projections with the usual ordering.

The following theorem is an immediate consequence of Theorem 3.5 and Lemma 3.6. Note that there is no assumption on the cardinality of $\mathfrak{A}$.

THEOREM 3.7. *Suppose $\mathfrak{A}$ is a C*-algebra and $\mathcal{P}(\mathfrak{A})$ contains an infinite chain. If the continuum hypothesis fails, there are nontrivial ultrafilters α, β on $\mathbb{N}$ such that $\mathfrak{A}^\alpha$ and $\mathfrak{A}^\beta$ are not $*$-isomorphic.*

COROLLARY 3.8. *Suppose $\mathfrak{A}$ is a C*-algebra with cardinality $2^{\aleph_0}$ such that $\mathcal{P}(\mathfrak{A})$ contains an infinite chain. The following are equivalent:*

1. *The continuum hypothesis holds.*
2. *For all nontrivial ultrafilters α, β on $\mathbb{N}$, $\mathfrak{A}^\alpha$ and $\mathfrak{A}^\beta$ are $*$-isomorphic.*

4. Embeddings into ultraproducts and finite local reflexivity

In the classical theory of ultraproducts, a theorem of S. Shelah [S1] states that two models $\mathfrak{A}$ and $\mathcal{B}$ have isomorphic ultrapowers if and only if they are elementarily equivalent (i.e., certain first-order statements are true in $\mathfrak{A}$ if and only if they are true in $\mathcal{B}$). A version of this result for isometric isomorphism of ultrapowers of Banach spaces was obtained by C. W. Henson [Hn1,2], and a version for homeomorphic linear isomorphism of ultrapowers of Banach spaces was obtained by S. Heinrich and C. W. Henson [HH]. The techniques of Henson [Hn1, 2] can be used to prove a similar result for $*$-isomorphism of ultrapowers of C*-algebras, but it is beyond the scope of this paper. However, the following result, which is a variant of Theorem 1.7 in [Hn2] is somewhat interesting. The proof, which is similar to that of [He; 6.3], is omitted. The gist of the following result is that the C*-algebra $\mathfrak{A}$ can be embedded in an ultrapower of $\mathcal{B}$ if and only if each finite "part" of $\mathfrak{A}$ can be approximately embedded in $\mathcal{B}$.

PROPOSITION 4.1. *Suppose $\mathfrak{A}$ and $\mathcal{B}$ are C^*-algebras. The following are equivalent:*

1. *$\mathfrak{A}$ is $*$-isomorphic to a subalgebra of an ultrapower of $\mathcal{B}$.*
2. *For any $X_1, \ldots, X_n$ in $\mathfrak{A}$, $*$-polynomials $p_1, \ldots, p_n$ in n variables and positive numbers $r_1, s_1, \ldots, r_n, s_n$ such that $r_k < \|p_k(X_1, \ldots, X_n)\| < s_k$ for $1 \le k \le n$, there are $Y_1, \ldots, Y_n$ in $\mathcal{B}$ such that $r_k < \|p_k(Y_1, \ldots, Y_n)\| < s_k$, for $1 \le k \le n$.*
3. *There is a net $\{\varphi_\lambda\}$ of maps whose domains form an increasingly directed collection of subsets of $\mathfrak{A}$ whose union $\mathcal{D}$ is a dense $*$-subalgebra (over the field $\mathbb{Q} + i\mathbb{Q}$) of $\mathfrak{A}$, and whose ranges are contained in $\mathcal{B}$ such that for all X, Y in $\mathcal{D}$ and all r, s in $\mathbb{Q} + i\mathbb{Q}$:*

$$\text{a. } \lim_\lambda \|\varphi_\lambda(rX + sY^*) - (r\varphi_\lambda(X) + s\varphi_\lambda(Y)^*)\| = 0,$$
$$\text{b. } \lim_\lambda \|\varphi_\lambda(XY) - \varphi_\lambda(X)\varphi_\lambda(Y)\| = 0,$$
$$\text{c. } \liminf_\lambda \|\varphi_\lambda(X)\| \ge \|X\|/2.$$

If $\mathfrak{A}$ and $\mathcal{B}$ in the preceding proposition are unital C*-algebras, and we want the $*$-isomorphism of $\mathfrak{A}$ into an ultrapower of $\mathcal{B}$ in (1) to be unital, then we change statement (2) by replacing X_1 by 1, and statement (3) by adding the conditions that $1 \in \mathcal{D}$ and $\|1 - \varphi_\lambda(1)\| \to 0$. In statement (2), it is sufficient to choose the X_k's from some generating set of $\mathfrak{A}$.

Whether every finite von Neumann algebra with a separable predual can be embedded into the ultrapower of the hyperfinite II_1 factor with respect to a free ultrafilter on $\mathbb{N}$ is an outstanding open question [C]. It has many deep connections with norms on tensor products (see [Ki]) and free entropy in free probablity theory [V]. As a consequence of the above proposition, we have the following corollary.

COROLLARY 4.2. *Suppose $\mathcal{M}$ and $\mathcal{N}$ are finite von Neumann algebras with faithful normal traces τ and σ, respectively. The following are equivalent:*

1. *There is a normal embedding π of $\mathcal{M}$ into some tracial ultrapower of $\mathcal{N}$ such that $\pi \circ \sigma = \tau$.*

2. *For every $n, N \in \mathbb{N}$, $\varepsilon > 0$, and every $A_1, \ldots, A_n$ in the unit ball of $\mathcal{M}$, there are elements $B_1, \ldots, B_n$ in the unit ball of $\mathcal{N}$ such that*

$$|\tau(A_{k_1} \cdots A_{k_s}) - \sigma(B_{k_1} \cdots B_{k_s})| < \varepsilon$$

for $1 \leq k_1, \ldots, k_s \leq n$ and $1 \leq s \leq N$.

It was proved by C. W. Henson, L. C. Moore, Jr. [HM] and J. Stern [St] (see [He; 6.7]) that if $\mathcal{Y}$ is a Banach space, there is an ultrapower $\mathcal{Y}^\alpha$ of $\mathcal{Y}$ and an isometric embedding π of the second dual $\mathcal{Y}^{\#\#}$ of $\mathcal{Y}$ into $\mathcal{Y}^\alpha$ so that π restricted to the natural embedding of $\mathcal{Y}$ in $\mathcal{Y}^{\#\#}$ induces the natural embedding of $\mathcal{Y}$ into $\mathcal{Y}^\alpha$. Moreover, there is a norm 1 projection mapping from $\mathcal{Y}^\alpha$ onto $\pi(\mathcal{Y}^{\#\#})$. This result is based on the principle of *local reflexivity* [LR], [JRZ], which says that if $\mathcal{Y}$ is a Banach space, $\mathcal{M}$ is a finite-dimensional subspace of $\mathcal{Y}^{\#\#}$ and $\mathcal{L}$ is a finite-dimensional subspace of $\mathcal{Y}^{\#}$, and $\varepsilon > 0$, then there is an operator $T: \mathcal{M} \to \mathcal{Y}$ that is a $(1+\varepsilon)$-isomorphism onto its range such that $T|\mathcal{M} \cap \mathcal{Y}$ is the identity map and, for each x in $\mathcal{M}$ and each f in $\mathcal{L}$, $f(Tx) = x(f)$.

If $\mathfrak{A}$ is a C*-algebra, then $\mathfrak{A}^{\#\#}$ is a von Neumann algebra, so it is natural to ask if the maps above can be chosen to be $*$-homomorphisms. However, this is not the case in general. For example, $C[0,1]$ is a commutative C*-algebra with no nontrivial projections, and thus any ultrapower $C[0,1]^\alpha$ has no nontrivial projections (see Proposition 2.1). But $C[0,1]^{\#\#}$ has many nontrivial projections. Hence there is no $*$-isomorphism of $C[0,1]^{\#\#}$ into an ultrapower of $C[0,1]$. In spite of the fact that the above embeddings cannot be made to be $*$-homomorphisms in the C*-algebra case, they can be chosen to be completely positive.

One conceivable way to prove the existence of a completely positive embedding of $\mathfrak{A}^{\#\#}$ into an ultrapower of $\mathfrak{A}$ would be to prove a completely positive version of local reflexivity for C*-algebras. However, Effros and Haagerup [EH] (see also [ER]) have shown that such a result is not generally true. Arveson [A] has shown that completely contractive unital maps are completely positive, so it follows that there is no completely contractive version of local reflexivity for operator spaces. However, we prove that there is a finitely contractive version.

An operator space is a closed linear subspace $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, along with the family $\{\| \ \|_n\}_{n=1}^{\infty}$ of norms, where each $\| \ \|_n$ is the norm on $M_n(\mathcal{S})$ as a subspace of $M_n(\mathcal{B}(\mathcal{H}))$. If $\mathcal{S}$ and $\mathcal{T}$ are operator spaces and $T: \mathcal{S} \to \mathcal{T}$ is linear, then $T_n: M_n(\mathcal{S}) \to M_n(\mathcal{T})$ is the map $T_n((S_{ij})) = (T(S_{ij}))$. We say T is *completely bounded* if $\|T\|_{\text{cb}} = \sup_{n \geq 1} \|T_n\|_n < \infty$. We say T is *completely contractive* if $\|T\|_{\text{cb}} \leq 1$. For background material on operator spaces we refer the reader to [Pi].

THEOREM 4.3. *Suppose $\mathcal{S}$ is an operator space, $\mathcal{M}$ is a finite-dimensional subspace of $\mathcal{S}^{\#\#}$, $\mathcal{L}$ is a finite-dimensional subspace of $\mathcal{S}^{\#}$, $\varepsilon > 0$, and n is a positive integer. Then there is a map $T: \mathcal{M} \to \mathcal{S}$ such that:*
1. *$T|(\mathcal{M} \cap \mathcal{S})$ is the identity map;*
2. *for each X in $\mathcal{M}$ and f in $\mathcal{L}$, $f(TX) = X(f)$;*
3. *$\|T_n\|, \|T_n^{-1}\| \leq 1 + \varepsilon$.*

Proof. Let $\delta > 0$. By replacing $\mathcal{L}$ with a larger finite-dimensional subspace, we can assume that, for every A in $M_n(\mathcal{M})$ with $\|A\| = 1$,

$$\sup\{|\langle A, F\rangle| : F \in M_n(\mathcal{L}), \|F\| = 1\} \geq 1 - \delta.$$

Using the fact that the Banach space $M_n(\mathcal{S})$ is locally reflexive, we obtain a linear map $\rho: M_n(\mathcal{M}) \to M_n(\mathcal{S})$ that is a $(1+\delta)$-isomorphism such that $\rho|M_n(\mathcal{M} \cap \mathcal{S})$ is the identity and such that, for each X in $M_n(\mathcal{M})$ and each f in $M_n(\mathcal{L})$, $f(\rho(X)) = X(f)$.

Let $\mathcal{G}$ be the group of all unitary $n \times n$ complex matrices such that each row has exactly one nonzero entry, and each nonzero entry is either 1 or -1. Define a mapping $\sigma: M_n(\mathcal{M}) \to M_n(\mathcal{S})$ by

$$\sigma(A) = \frac{1}{|\mathcal{G}|^2} \sum_{U, V \in \mathcal{G}} U^{-1} \rho(UAV) V^{-1}.$$

It is clear that $\|\sigma\| \leq \|\rho\| \leq 1 + \delta$, $\sigma|M_n(\mathcal{S} \cap \mathcal{M})$ is the identity map, and, for each X in $M_n(\mathcal{M})$ and each f in $M_n(\mathcal{L})$, $f(\sigma(X)) = X(f)$. It follows from the choice of $\mathcal{L}$ that, for each A in $M_n(\mathcal{M})$ with $\|A\| = 1$,

$$\|\sigma(A)\| \geq \sup\{|\langle A, F\rangle| : F \in M_n(\mathcal{L}), \|F\| = 1\} \geq 1 - \delta.$$

Hence $\|\sigma^{-1}\| \leq \frac{1}{1-\delta}$. We now specify δ so that $1+\delta, \frac{1}{1-\delta} \leq 1+\varepsilon$. Then $\|\sigma\|, \|\sigma^{-1}\| \leq 1 + \varepsilon$.

It follows from the definition of σ that $U^{-1}\sigma(UAV)V^{-1} = \sigma(A)$ for every A in $M_n(\mathcal{M})$ and every U, V in $\mathcal{G}$. Thus there must be a mapping $T: \mathcal{M} \to \mathcal{S}$ such that, for every matrix $(X_{ij}) \in M_n(\mathcal{M})$, $\sigma((X_{ij})) = (T(X_{ij}))$. In other words, $\sigma = T_n$ where $T_n: M_n(\mathcal{M}) \to M_n(\mathcal{S})$ is the canonical extension of T. It is clear that T satisfies (1) through (3) above.

□

COROLLARY 4.4. *If $\mathcal{S}$ is an operator space, then there are an infinite set $\mathbb{I}$ and a nontrivial ultrafilter α on $\mathbb{I}$, a completely isometric mapping $T: \mathcal{S}^{\#\#} \to \mathcal{S}^\alpha$, and a completely contractive surjective projection $E: \mathcal{S}^\alpha \to T(\mathcal{S}^{\#\#})$ such that $T|\mathcal{S}$ is the natural embedding. Moreover, if $\mathcal{S}$ is a C*-algebra, T and E are completely positive.*

Proof. Let $\mathbb{I}$ be the set of all 4-tuples $(\mathcal{M}, \mathcal{L}, \varepsilon, n)$ with $0 < \varepsilon < 1$, $n \in \mathbb{N}$, $\mathcal{M}$ a finite-dimensional subspace of $\mathcal{S}^{\#\#}$, and $\mathcal{L}$ a finite-dimensional subspace of $\mathcal{S}^{\#}$, and order $\mathbb{I}$ by $(\subset, \subset, >, <)$. For each $\iota = (\mathcal{M}, \mathcal{L}, \varepsilon, n)$ in $\mathbb{I}$, choose $T_\iota: \mathcal{M} \to \mathcal{S}$ satisfying (1)–(3) in Theorem 4.3.

Note that the collection of sets of the form $\{\iota \in \mathbb{I} : \iota \geq \iota_0\}$ (with ι_0 in $\mathbb{I}$), form a filter. Let α be an ultrafilter containing this filter. For each $\iota = (\mathcal{M}, \mathcal{L}, \varepsilon, n)$ in $\mathbb{I}$, define $\rho_\iota: \mathcal{S}^{\#\#} \to \mathcal{S}$ by $\rho_\iota|\mathcal{M} = T_\iota$, and $\rho|(\mathcal{S}^{\#\#} \setminus \mathcal{M}) = 0$. We define $T: \mathcal{S}^{\#\#} \to \mathcal{S}^\alpha$ by $T(x) = \{\rho_\iota(x)\}_\alpha$, we define $S: \mathcal{S}^\alpha \to T(\mathcal{S}^{\#\#})$ by $S(\{s_\iota\}_\alpha) = \lim_{\iota \to \alpha} s_\iota$ (limit is taken in weak $*$-topology), and we define $E = TS$. It is clear from the choice of the T_ι's that S is a left-inverse for T. The proof that T and E satisfy the required properties is an easy exercise from this point.

If $\mathcal{S}$ is a unital C*-algebra, then T, S, E are unital completely contractive maps, and thus, by Arveson's theorem [A], completely positive.

If $\mathfrak{A}$ is a nonunital C*-algebra, let $\mathcal{S}$ be the unitization of $\mathfrak{A}$. There is a unique state f on $\mathcal{S}$ whose kernel is $\mathfrak{A}$. We have $\mathfrak{A}^{\#\#}$ can be identified with $\{X \in \mathcal{S}^{\#\#} : X(f) = 0\}$. If, in the definition of $\mathbb{I}$, we assume, whenever $\iota = (\mathcal{M}, \mathcal{L}, \varepsilon, n)$, that $f \in \mathcal{L}$, then we have, for each $\iota \in \mathbb{I}$, $T_\iota(\mathcal{M} \cap \mathfrak{A}^{\#\#}) \subset \mathfrak{A}$. It follows that if T is defined for $\mathcal{S}$ as above, then $T(\mathfrak{A}^{\#\#}) \subset \mathfrak{A}$, and hence $E(\mathfrak{A}^\alpha) = T(\mathfrak{A}^{\#\#})$, so $T|\mathfrak{A}^{\#\#}$ and $E|\mathfrak{A}^\alpha$ are the required maps.

□

5. Maps on ultraproducts

Various maps on C*-algebras extend naturally to maps on ultraproducts.

Suppose $\{\mathfrak{A}_\iota : \iota \in \mathbb{I}\}$ and $\{\mathcal{B}_\iota : \iota \in \mathbb{I}\}$ are families of C*-algebras and, for each ι in $\mathbb{I}$, $\pi_\iota: \mathfrak{A}_\iota \to \mathcal{B}_\iota$ is a $*$-homomorphism, we define the ultraproduct $\pi = \prod^\alpha \pi_\iota$ of the π_ι's from $\prod^\alpha \mathfrak{A}_\iota$ into $\prod^\alpha \mathcal{B}_\iota$ by

$$\pi([\{A_\iota\}]) = [\{\pi_\iota(A_\iota)\}].$$

If each π_ι is a representation, i.e., $\mathcal{B}_\iota = \mathcal{B}(\mathcal{H}_\iota)$ for each ι in $\mathbb{I}$, then the ultraproduct π is a representation from $\prod^\alpha \mathfrak{A}_\iota$ into $\prod^\alpha \mathcal{B}(\mathcal{H}_\iota) \subseteq \mathcal{B}(\prod^\alpha \mathcal{H}_\iota)$.

If, for each ι in $\mathbb{I}$, $\varphi_\iota : \mathfrak{A}_\iota \to \mathcal{B}_\iota$ is a completely positive map, we define the ultraproduct $\varphi = \prod^\alpha \varphi_\iota$ in the same manner as $*$-homomorphisms. In particular, if each φ_ι is a state, then so is φ. The following proposition is an obvious consequence of the fact that if π is a $*$-homomorphism on a C*-algebra, then $\|\pi(X)\| = \operatorname{dist}(X, \ker \pi)$.

PROPOSITION 5.1. *Suppose $\{\mathfrak{A}_\iota : \iota \in \mathbb{I}\}$, $\{\mathcal{B}_\iota : \iota \in \mathbb{I}\}$, $\{\mathcal{J}_\iota : \iota \in \mathbb{I}\}$ are families of C*-algebras such that, for each ι in $\mathbb{I}$, $\mathcal{J}_\iota$ is a closed ideal in $\mathfrak{A}_\iota$, and $\pi_\iota : \mathfrak{A}_\iota \to \mathcal{B}_\iota$ is a $*$-homomorphism. Then, for any ultrafilter α on $\mathbb{I}$, $\ker(\prod^\alpha \pi_\iota) = \prod^\alpha \ker \pi_\iota$, and $\prod^\alpha \mathfrak{A}_\iota / \prod^\alpha \mathcal{J}_\iota = \prod^\alpha (\mathfrak{A}_\iota / \mathcal{J}_\iota)$.*

One of the consequences of the above proposition is that exact sequences of C*-algebras are preserved under ultraproduct construction.

The following theorem shows that ultraproducts with respect to nontrivial ultrafilters on $\mathbb{N}$ are usually highly nonseparable. Recall that $\prod_{n\in\mathbb{N}} \mathfrak{A}_n$, or simply $\prod \mathfrak{A}_n$, denotes the cartesian product of $\mathfrak{A}_n$'s, $\sum_{n\in\mathbb{N}} \mathfrak{A}_n$, or $\sum \mathfrak{A}_n$, denotes the subspace of $\prod \mathfrak{A}_n$ consisting of all sequences converging in norm to zero as n tends to infinity. We say that two elements A and B in a C*-algebra are *orthogonal* if $A^*B = AB^* = 0$.

THEOREM 5.2. *Suppose $\{\mathfrak{A}_n\}$ is a sequence of C*-algebras such that α and $\lim_{n\to\infty} \dim \mathfrak{A}_n = \infty$ is a nontrivial ultrafilter on $\mathbb{N}$. Then*

1. *there is a family $\{A_t : t \in [0,1]\}$ of positive elements in $\prod \mathfrak{A}_n$, $A_t = \{A_t(n)\}$, with $\|A_t(n)\| = 1$ for t in $[0,1]$ and n in $\mathbb{N}$, whose image in $\prod \mathfrak{A}_n / \sum \mathfrak{A}_n$ (and hence in $\prod^\alpha \mathfrak{A}_n$) is an orthogonal family;*
2. *there is no one-to-one continuous linear map from $\prod \mathfrak{A}_n / \sum \mathfrak{A}_n$ (or $\prod^\alpha \mathfrak{A}_n$) into $l^\infty(\mathbb{N})$;*
3. *$\prod \mathfrak{A}_n / \sum \mathfrak{A}_n$ and $\prod^\alpha \mathfrak{A}_n$ have no faithful continuous linear maps into $\mathcal{B}(l^2(\mathbb{N}))$;*
4. *if each $\mathfrak{A}_n$ is separable, then $\sum^\alpha \mathfrak{A}_n$ and $\sum \mathfrak{A}_n$ are not complemented in $\prod \mathfrak{A}_n$;*
5. *if each $\mathfrak{A}_n$ is unital and there is a positive integer N such that, for each n in $\mathbb{N}$, and each P in $\mathfrak{A}_n$, there are elements U_k, V_k $(1 \leq k \leq N)$ in $\mathfrak{A}_n$ with norm less than N such that $\sum_{k=1}^N U_k P V_k = 1$. Then every separable representation of $\prod \mathfrak{A}_n$ is a*

direct sum of representations that factor through the coordinate homomorphisms. In particular, $\prod \mathfrak{A}_n / \sum \mathfrak{A}_n$ *and* $\prod^\alpha \mathfrak{A}_n$ *have no non-zero separable representations;*

6. *if* $\mathfrak{A}$ *is a purely infinite C*-algebra, then* $\mathfrak{A}^\alpha$ *has no separable unital representation.*

Proof. 1. For each n in $\mathbb{N}$, we can choose a finite orthogonal set $\mathbb{E}_n$ of positive norm one elements in $\mathfrak{A}_n$ so that the limit of the cardinalities of the $\mathbb{E}_n$'s approaches ∞ as $n \to \infty$. In fact, if $\mathfrak{A}_n$ is infinite-dimensional, then $\mathfrak{A}_n$ must contain a hermitian element with infinite spectrum, and if $\mathcal{X}$ is a compact Hausdorff space with infinitely many points, $C(\mathcal{X})$ contains an infinite orthogonal set of norm-one positive elements. We can assume that each $\mathbb{E}_n$ is the finite set $\{0, 1, \ldots, k_n\}$ (each number corresponds to a norm one positive operator) and that $\lim_{n\to\infty} k_n = \infty$. For each t in $[0,1]$ and each n in $\mathbb{N}$, we define $A_t(n) = [tk_n]$ (where $[\ ]$ denotes the greatest integer function). Since $0 \le tk_n - [tk_n] < 1$, it follows, for every t in $[0,1]$, that $\lim_{n\to\infty} A_t(n)/k_n = t$. It is now clear that $\{n \in \mathbb{N} : A_s(n) = A_t(n)\}$ is finite whenever $0 \le s < t \le 1$. This implies the image of $\{A_t : t \in [0,1]\}$ in $\prod \mathfrak{A}_n / \sum \mathfrak{A}_n$ is orthogonal.

2. The linear span of the image of $\{A_t : t \in [0,1]\}$ in both $\prod \mathfrak{A}_n / \sum \mathfrak{A}_n$ and $\prod^\alpha \mathfrak{A}_n$ is isometrically isomorphic to the space $\mathcal{Y}$ of complex functions on $[0,1]$ having finite support with the supremum norm on $\mathcal{Y}$. It suffices to show that there is no continuous linear map $\varphi\colon \mathcal{Y} \to l^\infty$. If such a φ exists, there must be $k, n \in \mathbb{N}$ such that $\mathbb{E} = \{t \in [0,1] : |\varphi(\chi_{\{t\}})(n)| \ge 1/k\}$ is infinite. For each $t \in \mathbb{E}$, choose λ_t with $|\lambda_t| = 1$ so that $\varphi(\lambda_t \chi_{\{t\}}) \ge 1/k$. Any finite sum of distinct $\lambda_t \chi_{\{t\}}$'s has norm 1, while the image under φ of such sums is unbounded, violating the boundedness of φ.

Statements (3)–(6) follow from (2). □

From the ultraproduct construction, we saw that the ultraproduct $\prod^\alpha \mathcal{H}_\iota$ of a family $\{\mathcal{H}_\iota : \iota \in \mathbb{I}\}$ of Hilbert spaces is a Hilbert space. There is a natural way in which the C*-ultraproduct $\prod^\alpha \mathcal{B}(\mathcal{H}_\iota)$ can be viewed as a subalgebra of $\mathcal{B}(\prod^\alpha \mathcal{H}_\iota)$, defined by $[\{T_\iota\}][\{h_\iota\}] = [\{T_\iota h_\iota\}]$.

If x, y are vectors in a Hilbert space $\mathcal{H}$, we let $\omega_{x,y}$ denote the rank-one operator defined by $\omega_{x,y}(h) = \langle h, y\rangle x$ for all h in $\mathcal{H}$. It is clear that if $x, y \in \prod^\alpha \mathcal{H}_\iota$, $x = [\{x_\iota\}]$, $y = [\{y_\iota\}]$, then $\omega_{x,y} = [\{\omega_{x_\iota, y_\iota}\}] \in \prod^\alpha \mathcal{B}(\mathcal{H}_\iota)$. We let $\mathcal{K}(\mathcal{H})$ denote the algebra of compact operators on $\mathcal{H}$.

LEMMA 5.3. *If* $\{\mathcal{H}_\iota\}_{\iota\in\mathbb{I}}$ *is a family of Hilbert spaces and* α *is a nontrivial ultrafilter on* $\mathbb{I}$, *then* $\prod^\alpha \mathcal{K}(\mathcal{H}_\iota) \supseteq \mathcal{K}(\prod^\alpha \mathcal{H}_\iota)$.

Proof. Since $\prod^\alpha \mathcal{K}(\mathcal{H}_\iota)$ is a C*-algebra containing all of the rank-one operators on $\prod^\alpha \mathcal{H}_\iota$, the lemma is proved. □

We list some more basic properties of ultraproducts in the following theorem. The first result is an analogue of Kadison's transitivity theorem [K] for ultraproducts.

THEOREM 5.4. *Assume α is a nontrivial ultrafilter on $\mathbb{I}$.*

1. *Suppose, for each ι in $\mathbb{I}$, $\mathfrak{A}_\iota$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H}_\iota)$ and T in $\mathcal{B}(\prod^\alpha \mathcal{H}_\iota)$ is in the weak-operator closure of $\prod^\alpha \mathfrak{A}_\iota$, and suppose $\mathcal{H}_0$ is a separable subspace of $\prod^\alpha \mathcal{H}_\iota$. Then there is an A in $\prod^\alpha \mathfrak{A}_\iota$ such that $\|A\| \le \|T\|$, $A|\mathcal{H}_0 = T|\mathcal{H}_0$, and $A^*|\mathcal{H}_0 = T^*|\mathcal{H}_0$. Moreover, if T is hermitian, positive, or unitary, then A can be chosen to be the same.*
2. *Suppose, for each ι in $\mathbb{I}$, $\pi_\iota\colon \mathfrak{A}_\iota \to \mathcal{B}(\mathcal{H}_\iota)$ is a $*$-representation. Then $\prod^\alpha \pi_\iota$ is irreducible if and only if the π_ι's are eventually irreducible along α.*
3. *An ultraproduct of primitive C^*-algebras is primitive.*
4. *An ultraproduct of pure states is a pure state.*
5. *An ultraproduct of tracial states is a tracial state.*

Proof. 1. There is no harm in assuming that $\mathcal{H}_0$ reduces T, since the smallest subspace reducing T that contains $\mathcal{H}_0$ is also separable. Let $\{e_0, e_1, e_2, \ldots\}$ be an orthonormal basis for $\mathcal{H}_0$, and for each $n \ge 0$, write $e_n = [\{e_n(\iota)\}]$. We want to show that the $e_n(\iota)$'s can be chosen so that, for each ι in $\mathbb{I}$,

a. $\{e_n(\iota) : 0 \le n < \dim \mathcal{H}_\iota\}$ is orthonormal, and
b. $e_n(\iota) = 0$ when $n \ge \dim \mathcal{H}_\iota$.

There is no harm in assuming that $e_0(\iota) \ne 0$ for each ι in $\mathbb{I}$ for which $\dim \mathcal{H}_\iota > 0$. Since $\|e_0\| = 1$, e_0 is not affected if we replace each $e_0(\iota)$ with $e_n(\iota)/\|e_n(\iota)\|$. Thus we can assume that $e_0(\iota), \ldots, e_n(\iota)$ have been chosen for each ι in $\mathbb{I}$ in accordance with (a), (b) above. For each ι in $\mathbb{I}$, choose a vector $f_{n+1}(\iota)$ in $\mathcal{H}_\iota$ so that $f_{n+1}(\iota)$ is the normalized projection of $e_{n+1}(\iota)$ onto the orthogonal complement of the linear span of $\{e_0(\iota), e_1(\iota), \ldots, e_n(\iota)\}$, $f_{n+1}(\iota) = 0$ if $\dim \mathcal{H}_\iota = n+1$, and $f_{n+1}(\iota)$ is any unit vector perpendicular to $\{e_0(\iota), e_1(\iota), \ldots, e_n(\iota)\}$ otherwise. It is clear from the orthonormality of $\{e_1, e_2, \ldots, e_{n+1}\}$ that $e_{n+1} = [\{f_{n+1}(\iota)\}]$.

Let a_{jk} be $\langle Te_k, e_j\rangle$ for $0 \le j, k < \infty$. For each ι in $\mathbb{I}$, and each A in $\mathfrak{A}_\iota$, define

$$\delta_\iota(A) = \sum_{k=1}^{\infty} \frac{1}{2^k}\Big(\Big\|Ae_k(\iota) - \sum_{j=1}^{\infty} a_{jk}e_j(\iota)\Big\| + \Big\|A^*e_k(\iota) - \sum_{j=1}^{\infty} \bar{a}_{kj}e_j(\iota)\Big\|\Big),$$

and define $\delta_\iota = \inf\{\delta_\iota(A) : A \in \mathfrak{A}_\iota, \|A\| \le \|T\|\}$. It follows from the fact that T is in the $*$-strong closure of $\{S \in \prod^\alpha \mathfrak{A}_\iota : \|S\| \le \|T\|\}$, that $\lim_{\iota\to\alpha} \delta_\iota = 0$.

Next choose a decreasing sequence $\{\mathbb{I}_n\}$ of sets in α whose intersection is $\emptyset$, with $\mathbb{I}_1 = \mathbb{I}$. For each $n \geq 1$, and each ι in $\mathbb{I}_n \setminus \mathbb{I}_{n+1}$, choose an A_ι in $\mathfrak{A}_\iota$ such that $\delta_\iota(A_\iota) < \delta_\iota + \frac{1}{n}$. Let A be $[\{A_\iota\}]$ in $\prod^\alpha \mathfrak{A}_\iota$. Then it follows that $A|\mathcal{H}_0 = T|\mathcal{H}_0$ and $A^*|\mathcal{H}_0 = T^*|\mathcal{H}_0$.

2. Since α is an ultrafilter, either the π_ι's are eventually irreducible or eventually reducible. In the latter case we can choose, for each ι in $\mathbb{I}$, a projection P_ι commuting with π_ι such that $0 \neq P_\iota \neq 1$ whenever π_ι is reducible. It follows that $P = [\{P_\iota\}]$ is a nontrivial projection commuting with $\prod^\alpha \pi_\iota$.

Suppose that the π_ι's are eventually irreducible along α, and suppose e, f are unit vectors in $\prod^\alpha \mathcal{H}_\iota$, with $e = [\{e_\iota\}]$ and $f = [\{f_\iota\}]$. We can assume that π_ι is irreducible and $\|e_\iota\| = \|f_\iota\|$ for each ι in $\mathbb{I}$. It follows from Kadison's transitivity theorem [K] that, for each ι in $\mathbb{I}$, there is an A_ι in the closed unit ball of the range of π_ι such that $A_\iota e_\iota = f_\iota$. Choose $B = [\{B_\iota\}]$ in $\prod^\alpha \mathfrak{A}_\iota$ so that $(\prod^\alpha \pi_\iota)(B) = [\{A_\iota\}]$, which maps e to f. Hence $\prod^\alpha \pi_\iota$ is irreducible.

Statements (3) and (4) follow immediately from (2), and statement (5) is elementary.

□

COROLLARY 5.5. *An ultraproduct of simple C^*-algebras need not be simple.*

Proof. Suppose $\mathfrak{A}$ is a simple infinite-dimensional unital C*-algebra with a faithful tracial state τ (e.g., the reduced free group C*-algebras $C^*_{\mathrm{r}}(F_n)$). Let α be any nontrivial ultrafilter on $\mathbb{N}$ and choose a sequence $\{A_n\}$ of norm 1 positive elements of $\mathfrak{A}$ such that $\tau(A_n) \to 0$ as $n \to \infty$. Then $A = [\{A_n\}]$ is a nonzero positive element in $\mathfrak{A}^\alpha$ whose trace $\tau^\alpha(A) = 0$. Hence $\mathfrak{A}^\alpha$, having a non-faithful tracial state, is not simple.

□

There are some obvious conditions under which ultraproducts of simple algebras are simple.

PROPOSITION 5.6. *Suppose $\{\mathfrak{A}_\iota : \iota \in \mathbb{I}\}$ is a family of unital C^*-algebras and N is a positive integer such that, for every ι in $\mathbb{I}$ and every A in $\mathfrak{A}_\iota$ with $\|A\| = 1$, there are elements $B_1, C_1, \ldots, B_N, C_N$ in $\mathfrak{A}_\iota$ with norm at most N, such that $1 = \sum_{k=1}^N B_k A C_k$. Then $\prod^\alpha \mathfrak{A}_\iota$ is simple.*

COROLLARY 5.7. *If $\mathfrak{A}$ is the Calkin algebra (i.e., $\mathfrak{A} = \mathcal{B}(l^2(\mathbb{N}))/\mathcal{K}(l^2(\mathbb{N}))$), then for any ultrafilter α, $\mathfrak{A}^\alpha$ is simple.*

Proof. Suppose $A \in \mathfrak{A}$ and $\|A\| = 1$, and suppose $\varepsilon > 0$. We will show that there are elements X, Y in $\mathfrak{A}$ with $\|X\|, \|Y\| < 1 + \varepsilon$ such that $XAY = 1$. Replacing A with A^*A, we can assume that $0 \le A \le 1$. We can lift A to an operator, denoted again by A, in $\mathcal{B}(l^2(\mathbb{N}))$ with $0 \le A \le 1$. We can choose an isometry V whose range is the spectral projection of A of the interval $[\frac{1}{1+\varepsilon}, 1]$, such that V^*AV is an invertible operator whose norm is at most $1 + \varepsilon$. Let Y be the image in $\mathfrak{A}$ of V and X be the image in $\mathfrak{A}$ of $(V^*AV)^{-1}V^*$.

□

6. Ultraproducts on $\aleph_1$-complete ultrafilters

We end this paper with some remarks and results concerning different types of ultrafilters.

Recall that an ultrafilter α is *countably cofinal* if there is a sequence $\{A_n\}$ in α whose intersection is empty. Otherwise, α is called $\aleph_1$-*complete* [CN]. An infinite cardinal m is *Ulam measurable* if there is a nontrivial $\aleph_1$-complete ultrafilter on a set with cardinality m. Clearly, $\aleph_0$ is not Ulam measurable. More generally, if m is an infinite cardinal, an ultrafilter α is m-complete if α is closed under intersections of subcollections with cardinality less than m. An uncountable cardinal m is *measurable* if there is an ultrafilter α on a set with cardinality m, such that α is m-complete. Clearly, a measurable cardinal is Ulam measurable. On the other hand, S. Ulam proved (see [CN; 8.31]) that a cardinal is Ulam measurable if and only if it is not less than the smallest measurable cardinal. Thus the existence of nontrivial $\aleph_1$-complete ultrafilters is equivalent to the existence of Ulam measurable cardinals, which, in turn, is equivalent to the existence of measurable cardinals. It is possible that the existence of measurable cardinals is independent of the usual axioms of set theory (ZFC), but it has been shown that the proof of the consistency of the existence of measurable cardinals can not be accomplished using only (ZFC). For a nice discussion of these ideas see [CN].

Ultraproducts with respect to $\aleph_1$-complete ultrafilters are strange, and when the factors are countable or separable they are often trivial. We give some examples in the next proposition.

PROPOSITION 6.1. *Suppose α is a nontrivial ultrafilter on the set $\mathbb{I}$. Then*

1. *α is $\aleph_1$-complete if and only if there is no funtion $f\colon \mathbb{I} \to (0, \infty)$ such that* $\lim_{\iota \to \alpha} f(\iota) = 0$;

2. *if m is an uncountable cardinal, α is m-complete and $\mathcal{X}$ is a Banach space with a separating family $\mathcal{F}$ of linear functionals such that* $\mathrm{card}(\mathcal{F}) < m$, *then the natural inclusion of $\mathcal{X}$ into $\mathcal{X}^\alpha$ is onto*;

3. *if α and m are as in (2), and k is a cardinal less than m, $\{\mathcal{X}_\iota : \iota \in \mathbb{I}\}$ is a family of separable Banach spaces such that each $\mathcal{X}_\iota$ has a dense subset $\mathcal{D}_\iota$ such that* $\text{card}(\mathcal{D}_\iota) \leq k$, *then, eventually along α, all of the $\mathcal{X}_\iota$'s are isometrically isomorphisc to some Banach space $\mathcal{X}$, so that $\prod^\alpha \mathcal{X}_\iota$ is the same as $\mathcal{X}^\alpha$;*

4. *if α is $\aleph_1$-complete and $\{\mathcal{X}_\iota : \iota \in \mathbb{I}\}$ is a family of Banach spaces, then $\prod^\alpha \mathcal{X}_\iota$ is the same as the classical ultraproduct of the $\mathcal{X}_\iota$'s.*

Proof. 1. Suppose α is *not* $\aleph_1$-complete, then we can choose a countable decreasing sequence $\{\mathcal{A}_n\}$ in α whose intersection is empty. If we define $f\colon \mathbb{I} \to (0, \infty)$ by $f|(\mathcal{A}_n \setminus \mathcal{A}_{n+1}) = \frac{1}{n}$, then $\lim_{\iota\to\alpha} f(\iota) = 0$. On the other hand, suppose α is $\aleph_1$-complete, and suppose $f\colon \mathbb{I} \to (0, \infty)$ and $\lim_{\iota\to\alpha} f(\iota) = 0$. Then, for each positive integer n, $f^{-1}((0, \frac{1}{n}]) \in \alpha$. Since α is closed under countable intersections, $\emptyset = \cap_n f^{-1}((0, \frac{1}{n}]) \in \alpha$, a contradiction.

2. Suppose, for each ι in $\mathbb{I}$, $x_\iota \in \mathcal{X}$. It follows from (1) that, for each f in $\mathcal{F}$, $f(x_\iota)$ is eventually constant along α, and since $\text{card}(\mathcal{F}) < m$, and α is m-complete, there is a set $\mathcal{S}$ in α, such that for every $\iota, \jmath$ in $\mathcal{S}$, and every f in $\mathcal{F}$, $f(x_\iota) = f(x_\jmath)$, which implies that $x_\iota = x_\jmath$. Hence $\{x_\iota\}$ is eventually constant along α.

3. Let $\mathcal{K}$ be a set with $\text{card}(\mathcal{K}) = k$. For each ι in $\mathbb{I}$, we can find a bounded surjective linear mapping $T_\iota\colon l^1(\mathcal{K}) \to \mathcal{X}_\iota$ (i.e., map $\{\chi_{\{x\}} : x \in \mathcal{K}\}$ onto a dense subset of the closed unit ball of $\mathcal{X}_\iota$ and extend by linearity and continuity). Let $\mathcal{D}$ be the linear span of $\{\chi_{\{x\}} : x \in \mathcal{K}\}$ over the field $\mathbb{Q} + i\mathbb{Q}$. Then $\text{card}(\mathcal{D}) = k$, and, for each x in $\mathcal{D}$, it follows from (1) that $\|T_\iota x\|$ is eventually constant along α, and since $\text{card}(\mathcal{D}) < m$, there is a set $\mathcal{S}$ in α such that, for every $\iota, \jmath$ in $\mathcal{S}$, $\|T_\iota x\| = \|T_\jmath x\|$. It follows that, for every $\iota, \jmath$ in $\mathcal{S}$ and every x in $l^1(\mathcal{K})$, $\|T_\iota x\| = \|T_\jmath x\|$. Hence, for $\iota, \jmath$ in $\mathcal{S}$, the mapping $T_\iota x \to T_\jmath x$ defines an isometric linear isomorphism from $\mathcal{X}_\iota$ to $\mathcal{X}_\jmath$.

4. Suppose $\{x_\iota\}$ is in the Cartesian product of the $\mathcal{X}_\iota$'s (i.e., the x_ι's are not necessarily uniformly bounded). It follows from (1) that $\lim_{\iota\to\alpha} \|x_\iota\| \neq \infty$. Hence there is a bounded family $\{y_\iota\}$ such that $x_\iota = y_\iota$ is eventually along α. Furthermore, if $\|x_\iota\| \to 0$, then, by (1), $x_\iota = 0$ eventually along α. This proves (4).

□

REMARKS 6.2. 1. It is clear that the Banach space results in the preceding proposition also hold for C*-algebras. In the proof of (3), we can replace $l^1(\mathcal{K})$ with the universal C*-algebra with generators $\mathcal{K}$ subject to the conditions that $\|x\| \leq 1$ for every x in $\mathcal{K}$. Then the T_ι's are $*$-homomorphisms and the $\mathcal{X}_\iota$'s are eventually $*$-isomorphic along α.

2. By modifying the proofs of (2) and (3) in the preceding proposition, it can be shown that if α is $\aleph_m$-complete, $\mathcal{G}$ is a group with $\text{card}(\mathcal{G}) < m$, and $\{\mathcal{G}_\iota : \iota \in \mathbb{I}\}$

is a family of groups with $\sup\{\mathrm{card}(\mathcal{G}_\iota) : \iota \in \mathbb{I}\} < m$, then the natural embedding of $\mathcal{G}$ into the classical ultrapower $\mathcal{G}^\alpha$ is onto, and the $\mathcal{G}_\iota$'s are eventually, along α, isomorphic to one another. The role of $l^1(\mathcal{K})$ in the proof is played by the free group with k generators.

We have seen (Theorem 5.2) that ultrapowers of a separable infinite-dimensional C*-algebra (or, an infinite-dimensional Banach space) with respect to countably cofinal ultrafilters are always highly non-separable. Thus we can express the existence of measurable cardinals in terms of ultrapowers of C*-algebras.

Corolllary 6.3. *The following are equivalent*:
1. *There exists a measurable cardinal.*
2. *There is a free ultrafilter α and a separable infinite-dimensional C^*-algebra $\mathfrak{A}$ such that $\mathfrak{A}^\alpha$ is separable.*

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