THE HUREWICZ THEOREM

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1. Introduction

Let L be a simplicial complex, $\pi_n(L)$ its n^{th} homotopy group (relative to some base point) and $H_n(L)$ its n^{th} homology group. For each integer $n > 0$ let $h^* : \pi_n(L) \to H_n(L)$ be the Hurewicz homomorphism. Then the *Hurewicz theorem* states (see [2]):

(a) $h_* : \pi_1(L) \to H_1(L)$ *is onto and has the commutator subgroup* $[\pi_1(L), \pi_1(L)]$ *as kernel.*

(b) *if* $\pi_i(L) = 0$ *for* $1 \leqq i \leqq n$, *then* $h_* : \pi_{n+1}(L) \to H_{n+1}(L)$ *is an isomorphism and* $h_* : \pi_{n+2}(L) \to H_{n+2}(L)$ *is onto.*

The usual definitions of the homotopy groups of L only involve its underlying topological space and disregard the simplicial structure of L; consequently the corresponding proofs of the Hurewicz theorem are also of a topological nature. In [3] a definition of the homotopy groups of L and of the Hurewicz homomorphisms was given in terms of simplicial structure of L only. The object of this paper is, starting from these definitions to give a completely combinatorial proof of the Hurewicz theorem. In fact it will be shown that the Hurewicz theorem may be considered as a special case of a purely group theoretical theorem.

We shall only consider the case of a c.s.s. complex which has only one 0-simplex. This is no real restriction as every simplicial complex may be converted into a c.s.s. complex by a (partial ordering of its vertices and as every connected c.s.s. complex is of the same homotopy type as one which has only one 0-simplex.

The paper is divided into two parts. In Part I the necessary definitions are given and the Hurewicz theorem is formulated and reduced to a purely group theoretical theorem. The proof of this theorem is given in Part II.

PART I

2. C.s.s. complexes and c.s.s. groups

A *c.s.s. complex* K (see [1]) is a collection of elements (called *simplices*) to each of which is attached a *dimension* $n \leqq 0$, such that for every n-simplex $\sigma \in K$ and every integer i with $0 \leqq i \leqq n$ there are defined in K an $(n-1)$-simplex $\sigma\varepsilon^i$ (called *face*) and an $(n+1)$-simplex $\sigma\eta^i$ (called *degenerate*). The operators ε^i and η^i are required to satisfy the following identities

$$\begin{aligned}
\varepsilon^i\varepsilon^{j-1} &= \varepsilon^j\varepsilon^i & i &< j \\
\eta^{j-1}\eta^i &= \eta^i\eta^j & i &< j \\
\eta^j\varepsilon^i &= \varepsilon^i\eta^{j-1} & i &< j \\
\eta^j\varepsilon^i &= \text{identity} & i &= j, j+1 \\
\eta^j\varepsilon^i &= \varepsilon^{i-1}\eta^j & i &> j+1.
\end{aligned}$$

The set of the n-simplices of K is denoted by K_n. The face and degeneracy operators ε^i and η^i thus may be considered as functions $\varepsilon^i : K_n \to K_{n-1}$ and $\eta^i : K_n \to K_{n+1}$.

A c.s.s. map $f : K \to L$ is a dimension preserving function which commutes with all face and degeneracy operators, i.e., for every simplex $\sigma \in K_n$ and integer i with $0 \leqq i \leqq n$

$$(f\sigma)\varepsilon^i = f(\sigma\varepsilon^i)$$

$$(f\sigma)\eta^i = f(\sigma\eta^i).$$

A *c.s.s. group* G is a c.s.s. complex such that for every integer $n \geqq 0$

(a) G_n is a group.

(b) all face and degeneracy operations $\varepsilon^i : G_n \to G_{n-1}$ and $\eta^i : G_n \to G_{n+1}$ are homomorphisms.

Let G and H be c.s.s. groups. A *c.s.s. homomorphism* $f : G \to H$ is a c.s.s. map such that for every integer $n \geqq 0$ the restriction $f_n : G_n \to H_n$ is a homomorphism.

A c.s.s. group G is called *free* if G_n is a free (non abelian) group for all n.

Let G be a c.s.s. group. Define[1] for each integer $n \geqq 0$ a subgroup $\tilde{G}_n \subset G_n$ by

$$\tilde{G}_n = \cap_{i=1}^{n} \text{kernel } \varepsilon^i.$$

Then $\sigma \in \tilde{G}_{n+1}$ implies $\sigma\varepsilon^0 \in \tilde{G}_n$. Hence we may define a homomorphism $\tilde{\partial}_{n+1} : \tilde{G}_{n+1} \to \tilde{G}_n$ by

$$\tilde{\partial}_{n+1}\, \sigma = \sigma\varepsilon^0 \qquad \sigma \in \tilde{G}_{n+1}$$

For each integer $m < 0$ let $\tilde{G}_m = 1$ and let $\tilde{\partial}_{m+1} : \tilde{G}_{m+1} \to \tilde{G}_m$ be the trivial map. Then it can be shown that image $\tilde{\partial}_{n+1}$ is a normal subgroup of kernel $\tilde{\partial}_n$ for all n, i.e., $\tilde{G} = \{\tilde{G}_n, \tilde{\partial}_n\}$ is a (not necessarily abelian) chain complex. Its homology groups are

$$H_n(\tilde{G}) = \text{kernel } \tilde{\partial}_n / \text{image } \tilde{\partial}_{n+1}.$$

Let $\sigma \in$ kernel $\tilde{\partial}_n$. Then the element of $H_n(\tilde{G})$ containing σ will be denoted by $\{\sigma\}$.

3. The homotopy groups

Let K be a c.s.s. complex which has only one 0-simplex. Then we define a c.s.s. group G as follows. G_n is the (not necessarily abelian) group which has a generator $\bar{\sigma}$ for every $\sigma \in K_{n+1}$ and a relation $\overline{\tau\eta^0} = 1$ for every $\tau \in K_n$. As clearly the groups G_n are free, it suffices to define the face and degeneracy homomorphisms $\varepsilon^i : G_n \to G_{n-1}$ and $\eta^i : G_n \to G_{n+1}$ on the generators of G_n. This is done by the following formulas:

$$\bar{\sigma}\varepsilon^0 = (\overline{\sigma\varepsilon^0})^{-1}\, \overline{\sigma\varepsilon^1}$$

$$\bar{\sigma}\varepsilon^i = \overline{\sigma\varepsilon^{i+1}} \qquad 0 < i \leqq n$$

$$\bar{\sigma}\eta^i = \overline{\sigma\eta^{i+1}} \qquad 0 \leqq i \leqq n.$$

[1] This construction is due to J. C. Moore.

For every integer $n > 0$ we now define $\pi_n(K)$, *the* n^{th} *homotopy group of* K, by

$$\pi_n(K) = H_{n-1}(\tilde{G}).$$

4. The homology groups

We define a c.s.s. group A as follows. For each integer $n \geq 0$ let

$$A_n = G_n/[G_n, G_n]$$

where $[G_n, G_n]$ denotes the commutator subgroup of G_n, and let the face and degeneracy homomorphisms $\varepsilon^i : A_n \to A_{n-1}$ and $\eta^i : A_n \to A_{n+1}$ be those induced by the corresponding homomorphisms of G. Thus A is "*G made abelian*" and we write

$$A \to G/[G, G].$$

Foı each integer $n > 0$ we define $H_n(K)$, *the* n^{th} *homology group of* K, by

$$H_n(K) = H_{n-1}(\tilde{A}).$$

5. The Hurewicz homomorphisms

Let $k : G \to A$ denote the projection, i.e., k maps an n-simplex of G on the coset of $[G_n, G_n]$ containing it. Clearly k is a c.s.s. homomorphism. It induces a chain map $\tilde{k} : \tilde{G} \to \tilde{A}$ (i.e., $\tilde{\partial}_n \tilde{k} \sigma = \tilde{k} \tilde{\partial}_n \sigma$ for every $\sigma \in \tilde{G}_n$) and hence induces homomorphisms

$$\tilde{k}_* : H_{n-1}(\tilde{G}) \to H_{n-1}(\tilde{A})$$

for each integer $n > 0$.

For each integer $n > 0$ we now define the *Hurewicz homomorphism* $h_* : \pi_n(K) \to H_n(K)$ by

$$h_* = \tilde{k}_*.$$

6. The Hurewicz theorem and its reduction to a group theoretical theorem

We first formulate both halves of the *Hurewicz theorem* in Theorem 1a and 1b below.

THEOREM 1a. *Let K be a c.s.s. complex which has only one 0-simplex. Then the homomorphism $h_* : \pi_1(K) \to H_1(K)$ is onto and has $[\pi_1(K), \pi_1(K)]$ as kernel.*

THEOREM 1b. *Let K be a c.s.s. complex which has only one 0-simplex and let $\pi_i(K) = 0$ for $0 < i \leq n$. Then $h_* : \pi_{n+1}(K) \to H_{n+1}(K)$ is an isomorphism and $h_* : \pi_{n+2}(K) \to H_{n+2}(K)$ is onto.*

It follows immediately from the definition of the Hurewicz homomorphism (see §5) that Theorem 1a and 1b are a special case of the following group theoretical theorems.

THEOREM 2a. *Let F be a free c.s.s. group, let $B = F/[F, F]$ and let $\tilde{l} : \tilde{F} \to \tilde{B}$ be the chain map induced by the projection $l : F \to B$. Then $\tilde{l}_* : H_0(\tilde{F}) \to H_0(\tilde{B})$ is onto and has $[H_0(\tilde{F}), H_0(\tilde{F})]$ as kernel.*

THEOREM 2b. *Let F be a free c.s.s. group, let $B = F/[F, F]$ and let $\tilde{l} : \tilde{F} \to \tilde{B}$ be* *the chain map induced by the projection* $l: F \to B$. *Let* $H_i(\tilde{F}) = 0$ *for* $0 \leqq i < n$ *Then* $\tilde{l}_* : H_n(\tilde{F}) \to H_n(\tilde{B})$ *is an isomorphism into and* $\tilde{l}_* : H_{n+1}(\tilde{F}) \to H_{n+1}(\tilde{B})$ *is onto.*

PART II

7. Proof of Theorem 2a

The following lemmas will be needed for the proof of Theorem 2a.

LEMMA 1.[2] *Let* F *be a c.s.s. group and let* $\alpha_1, \cdots, \alpha_n \in F_{n-1}$ *be such that* $\alpha_i \varepsilon^{j-1} = \alpha_j \varepsilon_i$ *for* $0 < i < j \leqq n$. *Then there exists an* $\alpha \in F_n$ *such that* $\alpha\varepsilon^i = \alpha_i$ *for* $i = 1, \cdots, n$.

PROOF. Let $\beta_n = \alpha_n \eta^{n-1}$. Then $\beta_n \varepsilon^n = \alpha_n$. Now suppose that $\beta_{k+1} \in F_n$ already has been defined such that $\beta_{k+1}\varepsilon^i = \alpha_i$ for $i \geqq k+1$. Define

$$\beta_k = (\alpha_k \eta^{k-1})(\beta_{k+1}{}^{-1}\, \varepsilon^k \eta^{k-1})\beta_{k+1}.$$

Then

$$\begin{aligned} \beta_k \varepsilon^k &= (\alpha_k \eta^{k-1} \varepsilon^k)(\beta_{k+1}{}^{-1} \varepsilon^k \eta^{k-1} \varepsilon^k)(\beta_{k+1}\varepsilon^k) = \alpha_k \\ \beta_k \varepsilon^i &= (\alpha_k \eta^{k-1} \varepsilon^i)(\beta_{k+1}{}^{-1} \varepsilon^k \eta^{k-1} \varepsilon^i)(\beta_{k+1}\varepsilon^i) \\ &= (\alpha_k \varepsilon^{i-1} \eta^{k-1})(\beta_{k+1}{}^{-1} \varepsilon^i \varepsilon^k \eta^{k-1})\alpha_i \\ &= (\alpha_i \varepsilon^k \eta^{k-1})(\alpha_i^{-1} \varepsilon^k \eta^{k-1})\, \alpha_i = \alpha_i \quad i \geqq k+1, \end{aligned}$$

i.e., $\beta_k \varepsilon^i = \alpha_i$ for $i \geqq k$. By induction on k we finally obtain $\alpha = \beta_1 \in F_n$ such that $\alpha\varepsilon^i = \beta_1\varepsilon^i = \alpha_i$ for $i = 1, \cdots, n$.

REMARK. In the above proof the element $\alpha \in F_n$ was obtained from the elements $\alpha_1, \cdots, \alpha_n \in F_{n-1}$ by application of the following operations only: ε^i, η^i, multiplication and taking inverses. We shall denote this element $\alpha \in F_n$ obtained from $\alpha_1, \cdots, \alpha_n$ in this specific way, by $e(\alpha_1, \cdots, \alpha_n)$. Clearly if $l: F \to B$ is a c.s.s. homomorphism, then $le(\alpha_1, \cdots, \alpha_n) = e(l\alpha_1, \cdots, l\alpha_n)$. Also if $\alpha_i = 1_{n-1}$, the unit element of F_{n-1}, for all i, then $e(\alpha_1, \cdots, \alpha_n) = 1_n$, the unit element of F_n.

LEMMA 2. *Let* F *be a c.s.s. group, let* $B = F/[F, F]$ *and let* $l: F \to B$ *be the projection. Let* $\psi \in \tilde{B}_n$. *Then there exists a* $\phi \in \tilde{F}_n$ *such that* $l\phi = \psi$.

PROOF. Clearly l is a c.s.s. homomorphism onto. Hence there exists an $\alpha \in F_n$ such that $l\alpha = \psi$. Let $\beta = e(\alpha\varepsilon^1, \cdots, \alpha\varepsilon^n)$. Because $l(\alpha\varepsilon^i) = (l\alpha)\varepsilon^i = \psi\varepsilon^i = 1_{n-1}$ for $i \neq 0$ it follows that $l\beta = le(\alpha\varepsilon^1, \cdots, \alpha\varepsilon^n) = e(l(\alpha\varepsilon^1), \cdots, l(\alpha\varepsilon^n)) = 1_n$. Let $\phi = \alpha\beta^{-1}$, then clearly $l\phi = l\alpha = \psi$ and $\phi\varepsilon^i = (\alpha\varepsilon^i)(\beta^{-1}\varepsilon^i) = 1_{n-1}$ for $i \neq 0$, q.e.d.

PROOF OF THEOREM 2a. The first part of Theorem 2a follows immediately from the fact that $l: F \to B$ is a c.s.s. homomorphism onto.

Let $\sigma \in F^0$ be such that $\{l\sigma\} = 0$, i.e., there exists a $\psi \in \tilde{B}_1$ such that $l\sigma = \psi\varepsilon^0$. Let $\phi \in \tilde{F}_1$ be such that $l\phi = \psi$ and let $\tau = (\phi^{-1}\varepsilon^0)\sigma$. Then $\{\sigma\} = \{\tau\}$. Furthermore $l\tau = (l\phi^{-1}\varepsilon^0)(l\sigma) = 1_0$. Hence $\tau \varepsilon [F_0, F_0]$ and $\{\sigma\} = \{\tau\} \in [H_0(\tilde{F}), H_0(\tilde{F})]$. As $H_0(\tilde{B})$ is abelian (because B_0 is abelian) it follows that the kernel of $\tilde{l}_* : H_0(\tilde{F}) \to H_0(\tilde{B})$ is exactly $[H_0(\tilde{F}), H_0(\tilde{F})]$. This completes the proof.

[2] This lemma is due to J. C. Moore.

8. Proof of Theorem 2b

The following lemmas will be needed.

LEMMA 3. *Let F be a c.s.s. group and let $\alpha \in F_n$ and $\phi \in \tilde{F}_1$ be such that $\alpha\varepsilon^n \cdots \varepsilon^1 = \phi\varepsilon^0$. Then there exist elements $\beta_0, \cdots, \beta_n \in F_{n+1}$ such that*

$$\beta_0\varepsilon^0 = \alpha$$

$$\beta_i\varepsilon^i = \beta_{i-1}\varepsilon^i \quad 0 < i \leqq n$$

$$\beta_i\varepsilon^{n+1} \cdots \varepsilon^{i+1} = 1_i \quad 0 \leqq i \leqq n$$

PROOF. Let

$$\beta_0 = (\alpha\eta^0)(\alpha^{-1}\varepsilon^n \cdots \varepsilon^1\eta^0 \cdots \eta^n)(\phi\eta^1 \cdots \eta^n).$$

Then

$$\beta_0\varepsilon^0 = \alpha(\alpha^{-1}\varepsilon^n \cdots \varepsilon^1\eta^0 \cdots \eta^{n-1})(\phi\varepsilon^0\eta^0 \cdots \eta^{n-1}) =$$

$$\beta_0\varepsilon^{n+1} \cdots \varepsilon^1 = (\alpha\varepsilon^n \cdots \varepsilon^1)(\alpha^{-1}\varepsilon^n \cdots \varepsilon^1)(\phi\varepsilon^1) = 1_0.$$

Now suppose β_{k-1} has already been defined in such a manner that $\beta_{k-1}\varepsilon^{k-1} = \beta_{k-2}\varepsilon^{k-1}$ and $\beta_{k-1}\varepsilon^{n+1} \cdots \varepsilon^k = 1_{k-1}$.

Let

$$\beta_k = (\beta_{k-1}\varepsilon_k\eta^k)(\beta_{k-1}{}^{-1}\varepsilon^{n+1} \cdots \varepsilon^{k+2}\varepsilon^k\eta^k \cdots \eta^n)(\beta_{k-1}\varepsilon^{n+1} \cdots \varepsilon^{k+2}\varepsilon^k\eta^{k-1}\eta^{k+1} \cdots \eta^n).$$

Then a straightforward computation yields

$$\beta_k\varepsilon^k = \beta_{k-1}\varepsilon^k$$

$$\beta_k\varepsilon^{n+1} \cdots \varepsilon^{k+1} = l_k.$$

The lemma now follows by induction on k.

LEMMA 4. *Let F be a c.s.s. group, let $\gamma \in$ kernel $\tilde{\partial}_n$ and let $\alpha \in F_n$ and $\phi \in \tilde{F}_1$ be such that $\alpha\varepsilon^n \cdots \varepsilon^1 = \varphi\varepsilon^0$. Then there exists a $\lambda \in \tilde{F}_{n+1} \cap [F_{n+1}, F_{n+1}]$ such that $\lambda\varepsilon^0 = \gamma\alpha\gamma^{-1}\alpha^{-1}$.*

PROOF. For each integer i with $0 \leqq i \leqq n$ let

$$\lambda_i = (\gamma\eta^i)\beta_i(\gamma^{-1}\eta^i)\beta_i^{-1} \in [F_{n+1}, F_{n+1}$$

where β_i is as in Lemma 3. Then

$$\lambda_0\varepsilon^0 = \gamma\alpha\gamma^{-1}\alpha^{-1}$$

$$\lambda_i\varepsilon^i = \gamma(\beta_i\varepsilon^i)\gamma^{-1}(\beta_i^{-1}\varepsilon^i)$$

$$\lambda_i\varepsilon^{i+1} = \gamma(\beta_i\varepsilon^{i+1})\gamma^{-1}(\beta_i^{-1}\varepsilon^{i+1}) = \gamma(\beta_{i+1}\varepsilon^{i+1})\,\gamma^{-1}(\beta_{i+1}^{-1}\varepsilon^{i+1}),\ i \neq n$$

$$\lambda_i\varepsilon^j = l_n \quad j \neq i \quad i+1$$

$$\lambda_n\varepsilon^{n+1} = l_n.$$

Let

$$\lambda = \textstyle\prod_{i=0}^n(\lambda_i)^{\varepsilon_1} \quad \text{where} \quad \varepsilon_i = (-1)^i.$$

Then it is readily verified that

$$\lambda \in \tilde{F}_{n+1} \cap [F_{n+1}, F_{n+1}]$$
$$\lambda\varepsilon^0 = \gamma\alpha\gamma^{-1}\alpha^{-1}, \qquad \text{q.e.d.}$$

LEMMA 5. *Let F be a c.s.s. group such that $H_0(\tilde{F}) = 0$. Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in F_n$ be such that $\alpha_1\varepsilon^i = \beta_1\varepsilon^i$ and $\alpha_2\varepsilon^i = \beta_2\varepsilon^i$ for all i. Then there exists a $\nu \in \tilde{F}_{n+1} \cap [\tilde{F}_{n+1}, F_{n+1}]$ such that*

$$\nu\varepsilon^0 = \alpha_1\alpha_2\alpha_1^{-1}\alpha_2^{-1}\beta_2\beta_1\beta_2^{-1}\beta_1^{-1}.$$

PROOF. Because $H_0(\tilde{F}) = 0$ it follows (using Lemma 4) that there exist elements $\lambda, \mu \in \tilde{F}_{n+1} \cap [F_{n+1}, F_{n+1}]$ such that

$$\lambda\varepsilon^0 = (\beta_2^{-1}\alpha_2)\alpha_1^{-1}(\alpha_2^{-1}\beta_2)\alpha_1$$
$$\mu\varepsilon^0 = (\beta_1^{-1}\alpha_1)\beta_2(\alpha_1^{-1}\beta_1)\beta_2^{-1}.$$

Let

$$\nu = (\alpha_1\eta^0)(\beta_2\eta^0)\lambda(\beta_2^{-1}\eta^0)(\alpha_1^{-1}\eta^0)(\beta_1\eta^0)\mu(\beta_1^{-1}\eta^0).$$

Then a direct computation yields

$$\nu\varepsilon^0 = \alpha_1\alpha_2\alpha_1^{-1}\alpha_2^{-1}\beta_2\beta_1\beta_2^{-1}\beta_1^{-1}$$
$$\nu\varepsilon^i = 1_n \qquad i \neq 0, \qquad \text{q.e.d.}$$

LEMMA 6. *Let F be a free c.s.s. group and let $H_i(\tilde{F}) = 0$ for $0 \leq i < n$. Then there exist homomorphisms $D_i : F_i \to F_{i+1}$ $(0 \leq i < n)$ such that for every $\alpha \in F_i$*

$$(D_i\alpha)\varepsilon^0 = \alpha$$
$$(D_i\alpha)\varepsilon^j = D_{i-1}(\alpha\varepsilon^{j-1}) \qquad j \neq 0.$$

PROOF. Let K be an integer such that $0 \leq k < n$ and suppose that for $i < k$ homomorphisms $D_i : F_i \to F_{i+1}$ have been defined satisfying the above conditions. As F_k is a free group it is sufficient to define D_k on a set of generators $\sum$ of F_k. This is done as follows. Let $\alpha \in \sum$ be a generator, and let

$$\delta = e(D_{k-1}(\alpha\varepsilon^0), \cdots, D_{k-1}(\alpha\varepsilon^k)).$$

Then for $0 \leq i \leq k$

$$(\alpha(\delta^{-1}\varepsilon^0))\varepsilon^i = (\alpha\varepsilon^i)(\delta^{-1}\varepsilon^0\varepsilon^i) = (\alpha\varepsilon_i)(\delta^{-1}\varepsilon^{i+1}\varepsilon^0)$$
$$= (\alpha\varepsilon^i)((D_{k-1}(\alpha^{-1}\varepsilon^i))\varepsilon^0) = (\alpha\varepsilon^i)(\alpha^{-1}\varepsilon^i) = l_{k-1}.$$

As $H_k(\tilde{F}) = 0$ there exists a $\phi \in \tilde{F}_{k+1}$ such that $\phi\varepsilon^0 = \alpha(\delta^{-1}\varepsilon^0)$. Now define

$$D_k\alpha = \phi\delta.$$

In order to prove that the homomorphism $D_k : F_k \to F_{k+1}$ defined in this manner has the desired properties it clearly suffices to show that this is the case for each generator $\alpha \in \sum$. Indeed for each $\alpha \in \sum$ we have

$$(D_k\alpha)\varepsilon^0 = (\phi\delta)\varepsilon^0 = \alpha(\delta^{-1}\varepsilon^0)(\delta\varepsilon^0) = \alpha$$
$$(D_k\alpha)\varepsilon^j = (\phi\delta)\varepsilon^j = \delta\varepsilon^j = D_{k-1}(\alpha\varepsilon^{j-1}) \quad j \neq 0.$$

The lemma now follows by induction on k.

LEMMA 7. *Let F be a free c.s.s. group and let $H_i(\bar{F}) = 0$ for $0 \leqq i < n$. Let $\rho \in$ kernel $\tilde{\partial}_n \cap [F_n, F_n]$. Then there exists a $\chi \in \bar{F}_{n+1} \cap [F_{n+1}, F_{n+1}]$ such that $\chi\varepsilon^0 = \rho$.*

PROOF. As $\rho \in [F_n, F_n]$ there exists an integer q and elements $\alpha_1, \cdots, \alpha_{2q} \in F_n$ such that

$$\rho = \textstyle\prod_{s=1}^{q} [\alpha_{2s-1}, \alpha_{2s}]$$

where [,] denotes the commutator. For $0 \leqq t \leqq 2q$ let

$$\delta_t = e(D_{n-1}(\alpha_t\varepsilon^0), \cdots, D_{n-1}(\alpha_t\varepsilon^n))$$

and let $\beta_t = \delta_t\varepsilon^0$. Then by Lemma 5 there exists for each integer s with $0 \leqq s \leqq q$ a $\nu_s \in F_{n+1} \cap [F_{n+1}, F_{n+1}]$ such that

$$\nu_s\varepsilon^0 = [\alpha_{2s-1}, \alpha_{2s}][\beta_{2s}, \beta_{2s-1}].$$

Let

$$\chi = \textstyle\prod_{s=1}^{q} (\nu_s[\delta_{2s-1}, \delta_{2s}]).$$

Then a direct computation yields that $\chi\varepsilon^0 = \rho$ and $\chi\varepsilon^i = 1_n$ for $i \neq 0$, q e.d.

PROOF OF THEOREM 2b. Let $\sigma \in$ kernel $\tilde{\partial} \cap F_n$ be such that $\{l\sigma\} = 0$, i.e., there exists a $\psi \in \bar{B}_{n+1}$ such that $l\sigma = \psi\varepsilon^0$. Let $\phi \in \bar{F}_{n+1}$ be such that $l\phi = \psi$ and let $\nu = (\phi^{-1}\varepsilon^0)\sigma$. Then $\{\sigma\} = \{\tau\}$ and $l\tau = (l\phi^{-1}\varepsilon^0)(l\sigma) = 1_n$, i.e., $\tau \in$ kernel $\tilde{\partial}_n \cap [F_n, F_n]$. Hence by Lemma 7 $\{\tau\} = 0$. This proves the first part of Theorem 2b.

Let $\xi \in$ kernel $\tilde{\partial}_{n+1} \cap B_{n+1}$. Then there exists a $\rho \in \bar{F}_{n+1}$ such that $l\rho = \xi$. As $\rho\varepsilon^0\varepsilon^i = \rho\varepsilon^{i+1}\varepsilon^0 = 1_{n-1}$ for all i and $l(\rho\varepsilon^0) = (l\rho)\varepsilon^0 = \xi\varepsilon^0 = 1_n$, it follows that $\rho\varepsilon^0 \in$ kernel $\tilde{\partial}_n \cap [F_n, F_n]$. By Lemma 7 there exists a $\chi \in \bar{F}_{n+1} \cap [F_{n+1}, F_{n+1}]$ such that $\chi\varepsilon^0 = \rho\varepsilon^0$. Hence $(\rho\chi^{-1})\varepsilon^i = 1_n$ for all i and $l(\rho\chi^{-1}) = l\rho = \xi$, i.e., $\bar{l}_* \{\rho\chi^{-1}\} = \{\xi\}$. This completes the proof.

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