DUALITY IN HOMOTOPY THEORY

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1. *Introduction.* Certain results ([7], [8], [10], [11]) suggest that there should be some principle of duality in homotopy theory. Among other things one is led to expect that cohomotopy groups will appear as dual to homotopy groups. But the fact that a cohomotopy group $\pi^n(X)$, unlike $\pi_n(X)$, is only defined if $\dim X \leqslant 2n-2$ is a serious obstacle to the formulation of such a principle. However, the set of S-maps (*i.e.* S-homotopy classes [11]) $X \to Y$ is a group for every pair of spaces X, Y. Therefore, this difficulty does not appear in S-theory [11].

In this paper we formulate a principle of duality in the S-theory of finite polyhedra. It is analogous to the Alexander duality in that it is primarily defined for subsets, in our case subpolyhedra, of (polyhedral) spheres. An "n-dual" of a subpolyhedron $X \subset S^n$ is a subpolyhedron $D_n X \subset S^n - X$ which is an "S-deformation retract" of $S^n - X$. An S-map $\alpha : X \to Y$, where Y is also a subpolyhedron of S^n, has a dual $D_n \alpha : D_n Y \to D_n X$, and the map $\alpha \to D_n \alpha$ is an isomorphism[1]

$$\{X,\ Y\} \approx \{D_n Y,\ D_n X\}.$$

If $D_n X$ is n-dual to X, then X is n-dual to $D_n X$ so we have an isomorphism of $\{D_n Y, D_n X\}$ onto $\{X, Y\}$. The duality is expressed by the statement that these two isomorphisms are inverse to each other. Among other things we show how the construction of a finite CW-complex by the successive attaching of cells can be dualized.

The main results are stated in sections 3 through 7. Section 2 is devoted to a summary of notation and background material and sections 8 through 13 contain the proofs of the results stated in the earlier sections.

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2. *Preliminaries.* Let R^∞ denote the subset of Hilbert space consisting of points $(t_0, t_1, \dots)$ such that $t_i = 0$ for all but a finite set of values of i. Let $v_n' = (\delta_0{}^n, \delta_1{}^n, \dots)$, where $\delta_i{}^n = 0$ if $i \neq n$, $\delta_n{}^n = 1$, and let $v_n = (-\delta_0{}^n, -\delta_1{}^n, \dots)$. Let $S_n{}^0 = v_n \cup v_n'$ and let S^n denote the geometrical join

$$S^n = S_0{}^0 * S_1{}^0 * \dots * S_n{}^0 = S^{n-1} * S_n{}^0.$$

If X, $Y \subset S^n$, we define $\{X, Y; n\}$ as the direct limit under suspension of the sets[1] $[S^k X, S^k Y]$, where

$$S^k W = W * S^0_{n+1} * \dots * S^0_{n+k} \quad (W = X,\ Y)$$

[1] We use $\{A, B\}$, $[A, B]$ to denote, respectively, the group of S-maps and the set of ordinary homotopy classes $A \to B$, where A, B are arbitrary spaces. If $f : A \to B$ denotes a map, then $\{f\}$, $[f]$ will denote the corresponding elements of $\{A, B\}$, $[A, B]$. We also denote an element $\alpha \in \{A, B\}$ by $\alpha : A \to B$. Greek letters, thus used, will always denote S-maps and italic letters, as in $f : A \to B$, will denote ordinary maps.

and v_i, v_i' are ordered as written. If $q \geqslant n$ an isomorphism

$$\theta(q, n) : \{X, Y; n\} \approx \{X, Y; q\}$$

is defined in the obvious way, and we define $\{X, Y\}$ as the direct limit of the groups $\{X, Y; n\}$, for every n such that $X, Y \subset S^n$, under the isomorphisms $\theta(q, n)$.

If $K \subset S^{n-1}$ is a simplicial complex, then $SK = K * S_n^0$ will denote the simplicial complex whose simplexes are $\sigma * v_n$, $\sigma * v_n'$ and their faces, for every simplex σ of K. In particular S^n has a rectilinear triangulation K_n, which is defined inductively by $K_n = SK_{n-1}$ if $n > 0$, K_0 being the unique triangulation of S^0.

By a *polyhedron* we shall mean the space covered by a finite rectilinear complex in R^∞. We shall sometimes use the same symbol to denote a polyhedron and one of its triangulations, always assumed to be rectilinear. A piecewise linear map $X \to Y$, where X, Y are polyhedra, is one which is simplicial with respect to some pair of (rectilinear) triangulations of X, Y. Notice that, since all our polyhedra derive their piecewise linear structure from R^∞, a polyhedron $A \subset X$ is necessarily a subpolyhedron of X [*i.e.* the inclusion map $A \subset X$ is piecewise linear ([**12**], Theorem 5)].

Let X denote an n-dimensional polyhedron and let $A = X \cap S^q$, where $q \geqslant 2n+1$. There is a piecewise linear homeomorphism $S^q \to S^q$, which maps A into a q-simplex of K_q [see (11.3) below]. Hence it follows that the inclusion map $A \subset S^q$ can be extended to a piecewise linear homeomorphism h, of X into S^q (see [2], p. 139, for the case $A = \varnothing$; the generalization to an arbitrary A presents no difficulty). Notice that, if B is any other polyhedron in S^q, of at most n dimensions, then X, A may be replaced by $X \cup B$, $A \cup B$ so as to avoid "accidental intersections" between hX and B [*i.e.* points in $hX \cap (B-A)$].

Since the reduced homology and cohomology groups[2] ([**6**], p. 18) suspend isomorphically and naturally, with respect to the homomorphisms induced by maps and their suspensions, it follows that, for any pair of spaces A, B, an S-map $\alpha : A \to B$ induces homomorphisms

$$\alpha_{\#} : H_q(A) \to H_q(B), \qquad \alpha^{\#} : H^q(B) \to H^q(A)$$

in the obvious way. If A, B are CW-complexes, then α is an *S-equivalence* (*i.e.* has a 2-sided inverse $\alpha^{-1} \varepsilon \{B, A\}$) if and only if $\alpha_{\#}$ is an isomorphism onto for each q. To see this let $f : S^k A \to S^k B$ denote a map representing α and assume that $S^k A$, $S^k B$ are 1-connected (as is certainly the case if $k \geqslant 3$ or if A, B are non-empty and $k \geqslant 2$). Then f is an ordinary homotopy equivalence ([**15**], Theorem 3) if and only if each $\alpha_{\#}$ is an isomorphism onto.

[2] All homology and cohomology groups occurring in the sequel will be reduced and will be denoted by $H_q(A)$, $H^q(A)$, etc.

Similar remarks apply to $\alpha^{\#}$ if the complexes A, B are finite[3]. Notice that, if A, B are 1-connected CW-complexes and if the S-homotopy class of a map $f: A \to B$ is an S-equivalence, then f is an ordinary homotopy equivalence.

Let $i: A \subset X$. Then we describe A as an *S-deformation retract* of X if and only if the *inclusion* S-map $\iota = \{i\}: A \subset X$ is an S-equivalence. In this case we describe $\iota^{-1}: X \to A$ as the *S-retraction by deformation*, or simply the *S-retraction*, of X on A. If $A' \subset A$ and any two of the inclusion S-maps $A' \subset A$, $A \subset X$, $A' \subset X$ are S-equivalences so obviously is the third.

Let K denote a triangulation of S^n and X a subcomplex of K. We describe X as *complete* if and only if every simplex of K with all its vertices in X belongs to X. Evidently the barycentric subdivision of any subcomplex is complete in the barycentric subdivision of K. The subcomplex $X^* \subset K$ *complementary* to X consists of the simplexes of K which do not meet X. Evidently X^* is complete and $(X^*)^* = X$ if X is complete. In the latter event X, X^* are deformation retracts (and hence S-deformation retracts) of $K - X^*$, $K - X$. Also in this case X is complete in SK and X, SX^*, likewise SX, X^*, are complementary to each other in SK.

Let X and also $A \subset X$ denote complete subcomplexes of K. Then $X^* \subset A^*$, where X^*, A^* are the subcomplexes of K complementary to X, A, and the diagram

$$\begin{array}{ccccc} H_p(A) & \xrightarrow{\iota_{\#}} & H_p(X) & & (\iota: A \subset X) \\ \mathfrak{D}_n \downarrow & & \downarrow \mathfrak{D}_n & & \\ H^{n-p-1}(A^*) & \xrightarrow{\iota_0^{\#}} & H^{n-p-1}(X^*) & & (\iota_0: X^* \subset A^*) \end{array} \tag{2.1}$$

is commutative [5], where each $\mathfrak{D}_n$ is the appropriate Alexander duality isomorphism. Therefore, if $\iota_{\#}$ is an isomorphism (onto), so is $\iota_0^{\#}$. Hence, if A is an S-deformation retract of X, so is X^* of A^*.

Let $X_1, \ldots, X_k$ denote polyhedra in S^n and for each $i = 1, \ldots, k$ let C_i denote a compact subset of $S^n - X_i$.

LEMMA (2.2). *There are polyhedra $X_1^*, \ldots, X_k^*$ such that*

$$C_i \subset X_i^* \subset S^n - X_i,$$

X_i^* is a deformation retract of $S^n - X_i$ and whenever $X_i \subset X_j$, then $X_j^* \subset X_i^*$.

Proof. Let K denote a triangulation of S^n such that $X_1, \ldots, X_k$ are covered by complete subcomplexes of K. Let X_i^* denote the subcomplex of K complementary to X_i. Then X_i^* is a deformation retract of $K - X_i$

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[3] See §6 of [14] or use the universal coefficient theorem for $H^q(C, S^k A)$, where C denotes a mapping cylinder for f.

and $X_j^* \subset X_i^*$ if $X_i \subset X_j$. Therefore the lemma follows on taking the mesh of K to be so small that $C_i \subset X_i^*$ for each $i = 1, \ldots, k$.

3. *n-duals.* Let X denote a polyhedron in S^n. By an *n-dual* of X we mean a polyhedron in $S^n - X$, which is an S-deformation retract of $S^n - X$. We shall use $D_n X$ to denote an n-dual of X. Evidently a polyhedron contained in $D_n X$ is an n-dual of X if, and only if, it is an S-deformation retract of $D_n X$. If X, X^* are complete, complementary subcomplexes of some triangulation of S^n, then each is a deformation retract of the complement of the other. Therefore X, X^* are mutually n-dual. We shall prove that X is n-dual to every $D_n X$. We first prove:

LEMMA (3.1). *If A is a polyhedron in X, which is an S-deformation retract of X, then every n-dual of X is an n-dual of A (whence $D_n X$ is an S-deformation retract of $D_n A$ if $D_n X \subset D_n A$).*

Proof. Let X^*, A^* be as in (2.1) and let the mesh of K, in §2, be so small that $D_n X \subset X^*$. Then $D_n X$ is an S-deformation retract of X^*, since $D_n X$, X^* are both n-dual to X. Also X^* is an S-deformation retract of A^* by a remark following (2.1). Therefore $D_n X$ is an S-deformation retract of A^*. Since A^* is n-dual to A, so is $D_n X$.

THEOREM (3.2). *If $D_n X$ is n-dual to X, then X is n-dual to $D_n X$. Furthermore, $D_n X$ is $(n+1)$-dual to SX (hence also $X = D_n D_n X$ to $SD_n X$). Thus we may set*

$$D_{n+1} X = SD_n X, \quad D_{n+1} SX = D_n X. \tag{3.3}$$

Proof. The first part follows from (3.1) with A, X, $D_n X$ replaced by $D_n X$, X^*, X. The second part follows from the fact that $D_n X$ is an S-deformation retract of X^*, which is complementary to SX in SK.

If X' is a polyhedron in $S^n - X$, then $S^n - X$ can be triangulated as a CW-complex with a subcomplex covering X'. Therefore X' is an n-dual of X if, and only if, $\iota_{\#} : H_q(X') \approx H_q(S^n - X)$ for every $q \geqslant 0$, where $\iota : X' \subset S^n - X$.

4. *The basic duality.* In §9 we define, for every pair of polyhedra X, $Y \subset S^n$ and every pair of n-duals $D_n X$, $D_n Y$, a map

$$D_n : \{X, Y\} \to \{D_n Y, D_n X\}$$

such that, if $\iota : X \subset Y$, $\iota' : D_n Y \subset D_n X$, then

$$D_n \iota = \iota', \tag{4.1}$$

and if $\alpha \varepsilon \{X, Y\}$, $\beta \varepsilon \{Y, Z\}$ (where Z is any polyhedron in S^n) then

$$D_n(\beta\alpha) = D_n(\alpha) D_n(\beta), \tag{4.2}$$

provided $D_n \alpha$, $D_n \beta$ are both relative to the same $D_n Y$. It follows from (4.1), (4.2) that, if α is an S-equivalence, so is $D_n \alpha$ and $D_n \alpha^{-1} = (D_n \alpha)^{-1}$. In particular, $D_n \iota^{-1} = \iota'^{-1}$ if X is an S-deformation retract of Y and ι, ι' are as in (4.1).

For any pairs of spaces A, B there is an isomorphism

$$S : \{A, B\} \approx \{SA, SB\}$$

such that corresponding elements are represented by the same map $S^k A \to S^k B$ $(k \geqslant 1)$. In conformity with (3.3) we shall prove that

$$\left.\begin{aligned} D_{n+1} &= SD_n : \{X, Y\} \to \{SD_n Y, SD_n X\} \\ D_{n+1} S &= D_n : \{X, Y\} \to \{D_n Y, D_n X\}. \end{aligned}\right\} \tag{4.3}$$

We shall also prove:

THEOREM (4.4). *The map D_n is uniquely determined by the conditions* (4.1), (4.2), (4.3).

Since X is n-dual to $D_n X$, by (3.2), we have $D_n : \{D_n Y, D_n X\} \to \{X, Y\}$. We shall prove:

THEOREM (4.5). $D_n D_n \alpha = \alpha$ *for every* $\alpha \varepsilon \{X, Y\}$ *or* $\{D_n Y, D_n X\}$.

In §12 we show that D_n is a homomorphism. Hence, and from (4.5), it follows that

$$D_n : \{X, Y\} \approx \{D_n Y, D_n X\}. \tag{4.6}$$

Let $D_n' W$ denote another n-dual of $W (= X, Y)$ and let $D_n'' W$ denote an n-dual of W containing $D_n W \cup D_n' W$ [such a $D_n'' W$ exists by (2.2)]. Let

$$\iota_W : D_n W \subset D_n'' W, \quad \iota'_W : D_n' W \subset D_n'' W$$

and let $\iota_W^0 : W \subset W$. Then we have homomorphisms

$$\{D_n' Y, D_n' X\} \xleftarrow{D_n'} \{X, Y\} \xrightarrow{D_n''} \{D_n'' Y, D_n'' X\}$$

and if $\alpha \varepsilon \{X, Y\}$ it follows from (4.1) and the remarks following (4.2) that the diagrams

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ {\scriptstyle \iota_X^0}\downarrow & & \uparrow{\scriptstyle \iota_Y^0} \\ X & \xrightarrow{\alpha} & Y \end{array} \qquad\qquad \begin{array}{ccc} D_n'' X & \xleftarrow{D_n'' \alpha} & D_n'' Y \\ {\scriptstyle \iota_X}\uparrow & & \downarrow{\scriptstyle \iota_Y^{-1}} \\ D_n X & \xleftarrow{D_n \alpha} & D_n Y \end{array}$$

are dual to each other. Since we have a similar situation with D_n replaced by D_n' it follows from (4.2) that

$$D_n'' \alpha = \iota_X D_n(\alpha) \iota_Y^{-1} = \iota_X' D_n'(\alpha) \iota_Y'^{-1}.$$

Therefore $D_n'(\alpha) = \iota_X'^{-1} \iota_X D_n(\alpha) \iota_Y^{-1} \iota_Y'$. Let $\iota_W'' : D_n'' W \subset S^n - W$. Then $\iota_W^{-1} \iota_W' = (\iota_W'' \iota_W)^{-1} (\iota_W'' \iota_W')$ and we have

$$D_n'(\alpha) = \iota_X'^{-1} \iota_X D_n(\alpha) \iota_Y^{-1} \iota_Y', \tag{4.7}$$

where now $\iota_W : D_n W \subset S^n - W$, $\iota_W' : D_n' W \subset S^n - W$.

For any space A and any $p \geqslant -1$ let

$$\Sigma_p(A) = \{S^p, A\}, \quad \Sigma^p(A) = \{A, S^p\}.$$

Let $n \geqslant p$ and let $S_1^{n-p-1} = S_{p+1}^0 * \ldots * S_n^0$ if $n > p$. Then $S^n = S^p * S_1^{n-p-1}$ and we may take $D_n S^p = S_1^{n-p-1}$. It follows from (4.6) that

$$\theta D_n : \Sigma_p(Y) \approx \Sigma^{n-p-1}(D_n Y) \tag{4.8}$$

where $\theta : \{D_n Y, S_1^{n-p-1}\} \approx \Sigma^{n-p-1}(D_n Y)$.

Let $\alpha \varepsilon \{X, Y\}$ and consider the diagram

$$\begin{array}{ccc} H_p(X) & \xrightarrow{\alpha_\#} & H_p(Y) \\ \mathfrak{D}_n \downarrow & & \downarrow \mathfrak{D}_n \\ H^{n-p-1}(D_n X) & \xrightarrow{(D_n \alpha)^\#} & H^{n-p-1}(D_n Y) \end{array} \tag{4.9}$$

where each $\mathfrak{D}_n$ is an Alexander duality isomorphism[4]. In §10 we prove:

THEOREM (4.10). *The diagram (4.9) is commutative.*

In the diagram

$$\begin{array}{ccc} \Sigma_p(Y) & \xrightarrow{\tau} & H_p(Y) \\ \theta D_n \downarrow & & \downarrow \mathfrak{D}_n \\ \Sigma^{n-p-1}(D_n Y) & \xrightarrow{\tau^*} & H^{n-p-1}(D_n Y) \end{array} \tag{4.11}$$

let τ, τ^* denote the homomorphisms defined by

$$\tau\alpha = \alpha_\# u, \quad \tau^* \theta\beta = \beta^\# \mathfrak{D}_n u,$$

where $\alpha : S^p \to Y$, $\beta : D_n Y \to S_1^{n-p-1}$ and u is a fixed generator of $H_p(S^p)$ $\left(\mathfrak{D}_n u \,\varepsilon\, H^{n-p-1}(S_1^{n-p-1})\right)$. It follows from (4.10) that

$$\tau^* \theta D_n \alpha = (D_n \alpha)^\# \mathfrak{D}_n u = \mathfrak{D}_n(\alpha_\# u) = \mathfrak{D}_n \tau\alpha.$$

Hence we have proved:

COROLLARY (4.12). *The diagram (4.11) is commutative.*

If A_1, A_2 are any spaces we define

$$A_1 \vee A_2 = (A_1 \times a_2) \cup (a_1 \times A_2) \subset A_1 \times A_2,$$

[4] *i.e.* $\mathfrak{D}_n$ is composed of the Alexander duality isomorphism $H_p(W) \approx H^{n-p-1}(S^n - W)$ and $\iota_W^\# : H^{n-p-1}(S^n - W) \approx H^{n-p-1}(D^n W)$ $(W = X, Y)$. It follows from (4.1) that (2.1) is a special case of (4.9).

where $a_1 \varepsilon A_1$, $a_2 \varepsilon A_2$ are what we call the *base points* for $A_1 \vee A_2$. We shall use (A_λ, a_μ) $(\lambda, \mu = 1, 2;\ \lambda \neq \mu)$ to denote $A_1 \times a_2$ or $a_1 \times A_2$ according as $\lambda = 1$ or 2. We describe sequences

$$X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} X_3, \quad X_1^* \xleftarrow{\alpha_1^*} X_2^* \xleftarrow{\alpha_2^*} X_3^*,$$

of S-maps between polyhedra in S^n, as *n-dual* to each other if, and only if, $X_j^* = D_n X_j$, $\alpha_\lambda^* = D_n \alpha_\lambda$ $(j = 1, 2, 3;\ \lambda = 1, 2)$.

Let P_1, P_2 denote non-empty, proper subpolyhedra of S^n and let P_λ^* denote an n-dual of P_λ $(\lambda = 1, 2)$. In §11 we prove:

THEOREM (4.13). *With a suitable choice of base points, $p_\lambda \varepsilon P_\lambda$, $p_\lambda^* \varepsilon P_\lambda^*$, there are piecewise linear homeomorphisms h, h', of $P_1 \vee P_2$, $P_1^* \vee P_2^*$ into S^n, such that, if $X_\lambda = h(P_\lambda, p_\mu)$, $X_\lambda^* = h'(P_\lambda^*, p_\mu^*)$ $(\lambda \neq \mu = 1, 2)$, then the sequences*

$$X_\lambda \xrightarrow{\iota_\lambda} X_1 \cup X_2 \xrightarrow{\rho_\lambda} X_\lambda, \quad X_\lambda^* \xleftarrow{\rho_\lambda'} X_1^* \cup X_2^* \xleftarrow{\iota_\lambda'} X_\lambda^*$$

are n-dual to each other, where ι_λ, ι_λ' are inclusions and ρ_λ, ρ_λ' are the S-homotopy classes of the retractions in which $X_\mu \to X_1 \cap X_2$, $X_\mu^* \to X_1^* \cap X_2^*$.

Let Y be $(k-1)$-connected and let $\dim X = p \leqslant 2k-2$ $(k \geqslant 1)$. Then $\{X, Y\}$ may be identified with $[X, Y]$ in such a way that $\{f\} = [f]$ for every map $f: X \to Y$ (see (7.2) in [11]). Since $S^n - X$ is $(n-p-2)$-connected, we may take $D_n X$ to be $(n-p-2)$-connected (*e.g.* $D_n X =$ the complementary complex to X in some triangulation of S^n). Then we may similarly identify $\{D_n Y, D_n X\}$ with $[D_n Y, D_n X]$ if $\dim D_n Y \leqslant 2(n-p-2)$. Hence it follows from (4.8) that

$$\theta D_n : \pi_p(Y) \approx \pi^{n-p-1}(D_n Y) \quad (p \leqslant 2k-2) \tag{4.14}$$

if $\dim D_n Y \leqslant 2(n-p-2)$, as is certainly the case if $n \geqslant 2p+4$.

5. *Functorial presentation.* In this section it is shown how the results stated in the last section give rise to a duality in functorial form. Let Σ denote an arbitrary collection of spaces. By the *S-category* of Σ we mean the category whose objects are the spaces in Σ and whose maps are all the S-maps between them. Let $\mathbf{C}$, $\mathbf{C}'$ denote S-categories. We describe a (contravariant) functor $T: \mathbf{C} \to \mathbf{C}'$ as *linear* if and only if the map

$$T: \{X, Y\} \to \{TY, TX\}$$

is a homomorphism for every pair of spaces X, Y in $\mathbf{C}$. T is called an *isomorphism* (*onto*), denoted by $T: \mathbf{C} \approx \mathbf{C}'$, if and only if the object function $X \to TX$ and the mapping function $\alpha \to T\alpha$ are both 1-1 correspondences.

Let $\mathbf{X}_n$, $\mathbf{U}_n$ denote, respectively, the S-categories of all polyhedra in S^n and of their complements in S^n.

THEOREM (5.1). *There is a linear, contravariant isomorphism*

$$C_n : \mathbf{X}_n \approx \mathbf{U}_n$$

such that, if $\iota : X \subset Y, \ \iota' : S^n - Y \subset S^n - X,$

then $C_n \iota = \iota'$ (*whence* $C_n X = S^n - X$) *for* $X, Y \in \mathbf{X}_n$.

Proof. We define $C_n X = S^n - X$. Let $\alpha : X \to Y$ denote a map in $\mathbf{X}_n$ and let $\iota_W : D_n W \subset S^n - W$ where $W = X$ or Y and $D_n W$ is an n-dual of W. Then we define

$$C_n \alpha = \iota_X D_n(\alpha) \iota_Y^{-1} : S^n - Y \to S^n - X.$$

It follows from (4.7) that $C_n \alpha$ does not depend on the choice of $D_n X$, $D_n Y$, and the rest of the proof is left to the reader.

6. *Weakly dual constructions.* This section contains a theorem which enables us to dualize a certain process, of which attaching a cell to a polyhedron or, more generally, a finite CW-complex is a special case. We first define a relation of " weak " duality between finite CW-complexes, which are not necessarily polyhedra.

Let X denote a finite CW-complex. Then X is of the same homotopy type as a polyhedron X_0 ([**15**], p. 239). Moreover, we may take $\dim X_0 = \dim X = p$, say, and $X_0 \subset S^{2p+1}$. In the diagrams

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ \xi \downarrow & & \downarrow \eta \\ X_0 & \xrightarrow{\alpha_0} & Y_0 \end{array} \qquad\qquad \begin{array}{ccc} X^* & \xleftarrow{\alpha^*} & Y^* \\ \xi^* \uparrow & & \uparrow \eta^* \\ D_n X_0 & \xleftarrow{D_n \alpha_0} & D_n Y_0 \end{array} \tag{6.1}$$

let W, W^* ($W = X, Y$) denote finite CW-complexes, let W_0, $D_n W_0$ denote mutually dual polyhedra in S^n and let ξ, η, ξ^*, η^* denote S-equivalences. Then we describe W, W^* as *weakly* n-dual to each other. If α is a given S-map we define first α_0 and then α^* so that the diagrams (6.1) are commutative. We shall sometimes write $W^* = D_n W$, $\alpha^* = D_n \alpha$, remembering that the operator D_n, when thus defined, depends on ξ, η, ξ^*, η^*. Since W_0 may be chosen so that $\dim W_0 = \dim W = m$, say, we can choose W^* (*e.g.* $W^* = D_n W_0 =$ a deformation retract of $S^n - W$) so that $\dim W^* \leqslant n$ and W^* is $(n-m-2)$-connected.

Let $\beta \in \{Y, Z\}$ where Z is a finite CW-complex. Then it follows from (4.2) that

$$D_n(\beta\alpha) = D_n(\alpha) D_n(\beta)$$

provided $D_n \alpha$, $D_n \beta$ are both defined in terms of the same η, η^*.

From (4.3) it follows that $D_{n+1} \alpha = S D_n \alpha$ and $D_{n+1} S\alpha = D_n \alpha$ if $D_{n+1} \alpha$ and $D_{n+1} S\alpha$ are defined in terms of $\xi, \eta, S\xi^*, S\eta^*$ and $S\xi, S\eta, \xi^*, \eta^*$, respectively.

It follows from (4.5) that $D_n D_n \beta = \beta$ where $\beta \varepsilon \{X, Y\}$ or $\{Y^*, X^*\}$ provided $D_n \beta$ is defined in terms of η^{*-1}, ξ^{*-1}, η^{-1}, ξ^{-1} if $\beta \varepsilon \{Y^*, X^*\}$.

Thus (4.2), (4.3), (4.5) are valid for weak duality if ξ, η, ξ^*, η^* are suitably chosen, which we shall always assume to be the case. We do not assume that (4.1) is necessarily valid for weak duality when $X \subset Y$ [*Cf.* (4.13) and (6.2) below]. However, we note that (4.1) is valid if $X = Y$, $X^* = Y^*$, $\xi = \eta$, $\xi^* = \eta^*$.

Let X, Y denote finite CW-complexes and $f: X \to Y$ a cellular map. Let $\hat{X}$ denote a cone with vertex v_0 and X as base ($X \subset \hat{X}$). Assuming that $\hat{X}$, Y are disjoint from each other let Z_f denote the CW-complex obtained from $\hat{X} \cup Y$ by identifying each $x \varepsilon X$ with $fx \varepsilon Y$. The points in Z_f will be represented by $y \varepsilon Y$ and (x, t), where $x \varepsilon X$, $t \varepsilon I$, $(x, 0) = fx$, $(x, 1) = v_0$. Let $SX = X * (v \cup v')$, where v, v' are ordered as written, and let points in SX be represented by (x, s), where $-1 \leqslant s \leqslant 1$, $(x, -1) = v$, $(x, 1) = v'$. Define $g: Z_f \to SX$ by $g(x, t) = (x, 2t-1)$, $gY = v$.

Let $p = \dim X$, $q = \dim Y$, $r = \dim Z_f = \max(p+1, q)$ and let $n \geqslant 2r+1$. Then there are finite CW-complexes X^*, Y^*, which are weakly n-dual to X, Y and are such that X^* is $(n-p-2)$-connected and $\dim Y^* \leqslant n$. Since $n \geqslant 2p+3$, whence $n \leqslant 2(n-p-1)-1$, we have $D_n\{f\} = \{f^*\}$ for some map $f^*: Y^* \to X^*$ ([11], §7), which we assume to be cellular. Let Z_{f^*} and $g': Z_{f^*} \to SY^*$ be defined in the same way as Z_f and g. Let $i: Y \subset Z_f$, $i': X^* \subset Z_{f^*}$. Then, writing $Z_f = Z$, $Z_{f^*} = Z^*$, we have

$$Y \xrightarrow{i} Z \xrightarrow{g} SX, \qquad SY^* \xleftarrow{g'} Z^* \xleftarrow{i'} X^*$$

and it follows from (3.3) that Y, SY^*, likewise SX, X^*, are weakly $(n+1)$-dual to each other. In §13 we prove:

THEOREM (6.2). *Z^* is weakly $(n+1)$-dual to Z ($n \geqslant 2\dim Z+1$) in such a way that*

$$D_{n+1}\{i\} = -\{g'\}, \quad D_{n+1}\{g\} = \{i'\}. \tag{6.3}$$

Let ξ, η, ξ^*, η^* be the S-equivalences used, as in (6.1), to define $D_n\{f\}$. Then in (6.2) it is to be understood that ξ, η, ξ^*, η^* are given and that $D_{n+1}\{i\}$ and $D_{n+1}\{g\}$ are defined in terms of η, $S\eta^*$ and $S\xi$, ξ^*, together with a pair of S-equivalences $\zeta: Z \to Z_0$, $\zeta^*: D_{n+1} Z_0 \to Z_0^*$, where Z_0 is a polyhedron in S^{n+1} [see (13.2) below]. On replacing ζ, ζ^* by $-\zeta$, ζ^* or ζ, $-\zeta^*$ we have a weak duality between $\{i\}$, $\{g\}$ and $\{g'\}$, $-\{i'\}$. If there is an S-equivalence $\theta: Z \to Z$ such that $\theta\{i\} = -\{i\}$, $\{g\}\theta = \{g\}$, then $\zeta\theta$, ζ^* determine a weak duality between $\{i\}$, $\{g\}$ and $\{g'\}$, $\{i'\}$. It follows from (13.1) below that this is the case if $2\{f\} = 0$, *e.g.* if f is constant. Hence one can use (6.2), with a constant $f: X_1 \to SX_2$, to prove a weak form of (4.13).

Theorem (6.2) can be used to dualize some of the constructions which are fundamental in combinatorial homotopy. Consider, for example, the process of attaching a $(p+1)$-cell to a finite CW-complex Y by a map $f: S^p \to Y$ $(fS^p \subset Y^p)$. The dual process consists of attaching $\hat{Y}^*$ to S^{n-p-1} by a "dual" map $f^*: Y^* \to S^{n-p-1}$. We may assume that $f^* Y^{*n-p-2}$ is a single point and the result is then a CW-complex, which is weakly $(n+1)$-dual to the one obtained by adjoining the $(p+1)$-cell to Y. Let Y be a finite, $(k-1)$-connected CW-complex of at most $2k-1$ dimensions which has been constructed by attaching cells one at a time. Assume that the isomorphism (4.14), for the appropriate values of n, p, has been calculated at each stage. Then this construction can be dualized to give a weak q-dual of Y for a sufficiently large value of q.

Examples. Let P^2 denote the real and M^4 the complex projective planes and let

$$P^{k+1} = S^{k-1} P^2, \quad M^{k+2} = S^{k-2} M^4 \quad (k \geqslant 4).$$

Then P^{k+1} is of the form Z_f where X, Y are k-spheres and $f: X \to Y$ is of degree ± 2. We may take X^*, Y^* to be $(n-k-1)$-spheres and f^* will then be of degree ± 2. Hence, with a suitable choice of f^*, $Z_{f^*} = P^{n-k}$. Similarly $M^{k+2} = Z_f$, $Z_{f^*} = M^{n-k}$, where Y is a k-sphere, X a $(k+1)$-sphere and f is essential. Thus we describe P^2, M^4 as self-dual "up to suspension" and it follows from (4.14) that

$$\pi_{k+i}(P^{k+1}) \approx \pi^{r-i}(P^r), \qquad \pi_{k+i}(M^{k+2}) \approx \pi^{r-i}(M^r) \tag{6.4}$$

provided $k \geqslant i+2$, $r \geqslant 2i+2$.

As another example, let

$$Q^{k+2} = S_1^{\,k} \cup S_2^{k+1} \cup e^{k+2}, \qquad \bar{Q}^{k+2} = S_0^{\,k} \cup e_0^{k+1} \cup e_0^{k+2},$$

where the spheres $S_1^{\,k}$, S_2^{k+1} have a single common point, e^{k+2} is a $(k+2)$-cell attached to $S_1^{\,k} \cup S_2^{k+1}$ by a map $S^{k+1} \to S_1^{\,k} \cup S_2^{k+1}$ which is essential in $S_1^{\,k}$ and of degree $\pm m$ over S_2^{k+1}, and e_0^{k+1}, e_0^{k+2} are attached to $S_0^{\,k}$ by maps which are, respectively, of degree $\pm m$ and essential. Then $\bar{Q}^{n-k}$ is weakly $(n+1)$-dual to Q^{k+2}. Therefore

$$\pi_{k+i}(Q^{k+2}) \approx \pi^{r-i}(\bar{Q}^r) \quad (k \geqslant i+2;\ r \geqslant 2i+2). \tag{6.5}$$

Weak duals of the other elementary $A_n^{\,2}$-polyhedra ([4]) can be constructed without difficulty. Therefore (6.2), (4.13) provide an effective method of constructing a weak q-dual of a given $A_n^{\,2}$-polyhedron $(q \geqslant 2n+5)$.

7. *Applications.* We present some consequences of the results stated in the preceding sections.

THEOREM (7.1). *The cohomotopy groups* $\pi^i(X)$ $(\dim X \leqslant 2i-2)$ *of a finite* CW*-complex* X *are finitely generated.*

This follows from Prop. 1 on p. 491 of [9] and (4.14) [or induction on the number of cells in X and the exactness of the cohomotopy sequence of $(X, X-e)$ where e is a principal (open) cell of X].

THEOREM (7.2). *Let $G_1, G_2, \ldots$ denote a sequence of finitely generated abelian groups of which all but a finite number are zero. Then, for a sufficiently large q, there is a (finite) polyhedron P such that $\pi^{q+i}(P) \approx G_i$ for every $i \geqslant 1$.*

Proof. Let $G_i = 0$ if $i > l \geqslant 0$, let $p \geqslant 2l+1$, and let X denote a p-dimensional polyhedron [16] such that $\pi_i(X) \approx G_{p-i}$ for $-\infty < i < p$ [$\pi_i(X) = 0$ if $i < 0$]. Let $k = p-l \geqslant l+1$. Then X is $(k-1)$-connected and

$$p-1 = k+l-1 \leqslant 2k-2.$$

Let $X \subset S^n$ and let P denote an n-dual of X, where $n \geqslant 2p+2$. Let $q = n-p-1$. Then $\dim P \leqslant n$, whence $\pi^j(P) = 0$ if $j > n$, and it follows from (4.14) that

$$\pi^{q+i}(P) \approx \pi_{p-i}(X) \approx G_i \quad (i \geqslant 1).$$

Let A, B denote finite CW-complexes and, for any integer l, let

$$\{A, B\}_l = \{S^l A, B\} \text{ or } \{A, S^{-l} B\}$$

according as $l > 0$ or $l \leqslant 0$. For q sufficiently large let $D_q A$, $D_q B$ denote weak q-duals of A, B. Then if $l > 0$ it follows from (3.3) that we may take

$$D_{q+l} S^l A = D_q A, \; D_{q+l} B = S^l D_q B.$$

From this and a similar observation if $l \leqslant 0$ it follows that

$$D_{q+|l|} : \{A, B\}_l \approx \{D_q B, D_q A\}_l.$$

Let $f: X \to Y$, $Z = Z_f$ and $g: Z \to SX$ be as in (6.2) and let $i: Y \subset Z$. Let Q denote any finite CW-complex. Then we have sequences

$$\ldots \to \{Q, X\}_l \xrightarrow{f_\#} \{Q, Y\}_l \xrightarrow{i_\#} \{Q, Z\}_l \xrightarrow{g_\#} \{Q, X\}_{l-1} \to \ldots, \tag{7.3}$$

$$\ldots \to \{Y, Q\}_l \xrightarrow{f^\#} \{X, Q\}_l \xrightarrow{g^\#} \{Z, Q\}_{l-1} \xrightarrow{i^\#} \{Y, Q\}_{l-1} \to \ldots, \tag{7.4}$$

where $f_\#$, $f^\#$, $i_\#$, etc. are the homomorphisms induced by f, i, g and their suspensions, composed with $S^{-1}: \{Q, SX\}_l \approx \{Q, X\}_{l-1}$ if $l > 0$ in the case of $g_\#$ and with $S: \{X, Q\}_l \approx \{SX, Q\}_{l-1}$ if $l \leqslant 0$ in the case of $g^\#$.

For n sufficiently large let Q^* denote a weak n-dual of Q and let (7.3)*, (7.4)* denote the sequences analogous to (7.3), (7.4) with f, Z, Q replaced by f^*, Z^*, Q^*, where f^*, Z^* mean the same as in §6. Then it follows from (6.2) and (4.3) that the diagram

$$\begin{tikzcd}
\{Q, X\}_l \arrow[r, "f_\#"] \arrow[d, "D_{n+l}"'] & \{Q, Y\}_l \arrow[r, "i_\#"] \arrow[d, "D_{n+l}"'] & \{Q, Z\}_l \arrow[r, "g_\#"] \arrow[d, "D_{n+l}"] & \{Q, X\}_{l-1} \arrow[d, "D_{n+l-1}"] \\
\{X^*, Q^*\}_l \arrow[r, "(f^*)^\#"] & \{Y^*, Q^*\}_l \arrow[r, "g'^\#"] & \{Z^*, Q^*\}_{l-1} \arrow[r, "i'^\#"] & \{X^*, Q^*\}_{l-1}
\end{tikzcd} \quad (l > 0),$$

where each D_q is a weak duality isomorphism, is commutative up to sign. The same applies when $l \leqslant 0$ except that the two left-hand isomorphisms are replaced by D_{n-l} and the two on the right by D_{n-l-1}. We express this result as:

THEOREM (7.5). *The sequences* (7.3), (7.4)*, *likewise* (7.3)*, (7.4), *are weakly dual to each other up to sign.*

The sequence (7.4) is isomorphic to the direct limit under suspension of exact sequences of a kind introduced in [**3**]. Therefore (7.4), and hence also (7.3)*, likewise (7.4)* and (7.3), are exact. In particular, if $Q = S^0$, then (7.3), (7.4) are, respectively, isomorphic to the S-homotopy and the S-cohomotopy sequence of the pair (Z, Y). [*N.B.*: $\Sigma^m(X) \approx \Sigma^{m+1}(SX) \approx \Sigma^{m+1}(Z, Y)$.]

8. *Polyhedral mapping cylinders.* Let X, Y denote polyhedra and $f: X \to Y$ a map such that $fx = x$ if $x \in X \cap Y$. By a *polyhedral mapping cylinder* for f we mean a polyhedron P, containing $X \cup Y$, of which Y is a deformation retract such that $ri \simeq f$, rel. $X \cap Y$, where $i: X \subset P$ and $r: P \to Y$ is a retraction. If P is a polyhedral mapping cylinder for f, so obviously is SP for Sf. If $X \cap Y = \varnothing$, then P may be described as a polyhedral mapping cylinder for $[f]$. Let $r = \max(\dim X + 1, \dim Y)$.

LEMMA (8.1). *There is a polyhedral mapping cylinder P, for f, such that* $\dim P \leqslant r$.

Proof. Assume $X \cup Y$ to be triangulated as a simplicial complex with subcomplexes K, L covering X, Y. Then there is a subdivision K_1 of K, in which no simplex of $K \cap L$ is subdivided, and a homotopy $f \simeq f_1$, rel. $K \cap L$, such that f_1 is simplicial with respect to K_1 and L ([**13**], p. 289). Let the vertices of K_1 be ordered and let $K_1 \cup L$ be imbedded as a subcomplex of a simplex σ^N. For every simplex σ of K_1 with vertices $a_0, \ldots, a_p$, ordered as written, let

$$P_\sigma = \textstyle\bigcup_{\lambda=0}^{p} a_0 \ldots a_\lambda f_1(a_\lambda) \ldots f_1(a_p)$$

(if $v_0, \ldots, v_q$ are vertices of σ^N, then $v_0 \ldots v_q$ denotes the smallest simplex of σ^N which contains them, even if they are not distinct). Let

$$P = L \cup \bigcup_{\sigma \in K_1} P_\sigma.$$

Clearly $K_1 \cup L \subset P$. Let $i: K_1 \subset P$. Then a simplicial retraction $r: P \to L$ such that $ri = f_1$, is defined by $ra = f_1 a$ for every vertex a of K_1. Also $\dim P \leqslant r$.

It remains to prove that L is a deformation retract of P. Let $x_0, \ldots, x_q$ denote the vertices of $K_1 - K_1 \cap L$, correctly ordered, and let $P(\lambda)$ denote the union of all the simplexes of P which do not contain any of

$x_{\lambda+1}, \ldots, x_q$. Then

$$L = P(-1) \subset \ldots \subset P(\lambda-1) \subset P(\lambda) \subset \ldots \subset P(q) = P.$$

If a is a vertex of $K_1 \cap P(\lambda)$ which comes after x_λ, then $a \varepsilon K_1 \cap L$ whence $f_1 a = a$. It follows that for $\lambda \geqslant 0$ every simplex of $P(\lambda)$ which contains x_λ is of the form

$$a_0 \ldots x_\lambda f_1(y_1) \ldots f_1(y_k)$$

where $a_0 \ldots x_\lambda y_1 \ldots y_k$ is a simplex of K_1. This is a face of

$$a_0 \ldots x_\lambda f_1(x_\lambda) f_1(y_1) \ldots f_1(y_k).$$

Therefore the link of x_λ in $P(\lambda)$ is the join of $f_1(x_\lambda)$ and the link of $x_\lambda f_1(x_\lambda)$. Hence it follows that there is a homotopy $g_t: P(\lambda) \to P(\lambda-1)$, rel. $P(\lambda-1)$, such that $g_0 = 1$, $g_1 P(\lambda) = P(\lambda-1)$ and $g_t(x_\lambda * \sigma) = x(t) * \sigma$ for every simplex σ in the link of x_λ, where $x(t)$ is the point which divides $x_\lambda f_1(x_\lambda)$ in the ratio $t:1-t$. Therefore $P(\lambda-1)$ is a deformation retract of $P(\lambda)$. Hence it follows from a downward induction on λ that L is a deformation retract of P and the proof is complete.

Notice that, if X, $Y \subset S^q$, where $q \geqslant 2r+1$, then we may assume that $P \subset S^q$.

9. *Proof of* (4.1), ..., (4.5). We prove the existence of a map $D_n: \{X, Y\} \to \{D_n Y, D_n X\}$ having properties (4.1) through (4.5) by stages. First a certain map $\Delta_n: [X, Y] \to \{D_n Y, D_n X\}$ is defined and this is used to define D_n. The map Δ_n is, in turn, defined first for the case $X \cap Y = \varnothing$ and then extended to the general case.

A. *Definition of* $\Delta_n: [X, Y] \to \{D_n Y, D_n X\}$ *when* $X \cap Y = \varnothing$.

Let X, Y denote polyhedra in S^n with $X \cap Y = \varnothing$. Let $D_n X$, $D_n Y$ denote arbitrary, but fixed, n-duals of X, Y and let $f: X \to Y$ be given. Let $P \subset S^q$ $(q \geqslant n)$ be a polyhedral mapping cylinder for $[f]$. It follows from (2.2) that there are q-duals X^*, P^*, Y^* of X, P, Y such that

$$P^* \subset X^* \cap Y^*, \; D_q W = S^{q-n} D_n W \subset W^* \quad (W = X, Y). \quad (9.1)$$

Then $D_q X$ is an S-deformation retract of X^* and, by (3.1), so is P^* of Y^*. Therefore we have

$$D_q X \xleftarrow{\rho} X^* \xleftarrow{\iota^*} P^* \xleftarrow{\rho^*} Y^* \xleftarrow{\iota} D_q Y, \quad (9.2)$$

where ι, ι^* are S-inclusions and ρ, ρ^* are S-retractions by deformation. We define

$$|X^*, P^*, Y^*| = S^{n-q}(\rho\iota^*\rho^*\iota) \varepsilon \{D_n Y, D_n X\}, \quad (9.3)$$

where $S^{q-n}: \{D_n Y, D_n X\} \approx \{D_q Y, D_q X\}$ and $S^{n-q} = (S^{q-n})^{-1}$. Notice that, if $D_q Y \subset P^*$, then $\rho^{*-1}\iota' = \iota$, where $\iota': D_q Y \subset P^*$ (*N.B.*:

$\rho^{*-1}: P^* \subset Y^*$). Therefore

$$\iota^* \rho^* \iota = \iota^* \iota' : D_q Y \subset X^*.$$

Similarly if $D_q Y \subset P^* \cap D_q X$, then

$$\rho \iota^* \rho^* \iota : D_q Y \subset D_q X. \tag{9.4}$$

Our immediate objective is to show that $|X^*, P^*, Y^*|$ depends only on $[f]$ and not on the choices involved in its definition.

Let $P_1 \subset S^q$ denote a polyhedral mapping cylinder for $[f]$ and let $X^\#$, $P_1{}^\#$, $Y^\#$ denote q-duals of X, P_1, Y which satisfy (9.1). Assume that $X^\# \subset X^*$, $P_1{}^\# \subset P^*$, $Y^\# \subset Y^*$ and consider the diagram

$$\begin{tikzcd} D_q X & X^\# \arrow[l, "\rho'"'] \arrow[d, "\iota_3"] & P_1{}^\# \arrow[l, "\iota^\#"'] \arrow[d, "\iota_2"] & Y^\# \arrow[l, "\rho^\#"'] \arrow[d, "\iota_1"] & D_q Y \arrow[l, "\iota'"'] \arrow[dl, "\iota"] \\ & X^* \arrow[ul, "\rho"] & P^* \arrow[l, "\iota^*"] & Y^* \arrow[l, "\rho^*"] & \end{tikzcd} \tag{9.5}$$

in which the top line is the analogue of (9.2) and ι_1, ι_2, ι_3 are inclusion S-maps. We have $\iota_1 \rho^{\#-1} = \rho^{*-1} \iota_2$, whence $\rho^* \iota_1 = \iota_2 \rho^\#$. Similarly $\rho' = \rho \iota_3$ and so the diagram is commutative. Therefore $\rho \iota^* \rho^* \iota = \rho' \iota^\# \rho^\# \iota'$ and, in this case,

$$|X^*, P^*, Y^*| = |X^\#, P_1{}^\#, Y^\#|. \tag{9.6}$$

(*a*) *Independence of the choice of X^*, P^*, Y^*.*

Suppose $X^\#$, $P^\#$, $Y^\#$ also satisfy (9.1) relative to P. It follows from (2.2) that there are q-duals X^{**}, P^{**}, Y^{**} such that $P^{**} \subset X^{**} \cap Y^{**}$ and $Z^{**} \supset Z^* \cup Z^\#$ for $Z = X, P, Y$. Therefore, from (9.6) with $P_1 = P$,

$$|X^*, P^*, Y^*| = |X^{**}, P^{**}, Y^{**}| = |X^\#, P^\#, Y^\#|$$

and we write $|X^*, P^*, Y^*| = \Delta_n(P, q)$.

(*b*) *Independence of the choice of q.*

Let $r > q$ and let $\Delta_n(P, r)$ be defined by (9.3) when (9.2) is replaced by its $(r-q)$-fold suspension. Then, obviously,

$$\Delta_n(P, r) = \Delta_n(P, q) = \Delta_n(P), \text{ say.}$$

(*c*) *Independence of the choice of P.*

Let $P_1 \subset S^p$ denote another polyhedral mapping cylinder for $[f]$. By (*b*) above we may take $p = q$ and if $P \subset P_1$ it follows from (2.2) that there are q-duals X^*, P^*, $P_1{}^\#$, Y^* as in (9.5) with $X^\# = X^*$, $Y^\# = Y^*$. Therefore $\Delta_n(P) = \Delta_n(P_1)$ in this case.

If P, P_1 are arbitrary let $P_2 \subset S^r$ $(r \geqslant q)$ denote a polyhedral mapping cylinder for $[f]$ (*Cf.* §2) such that $P_2 \cap P_k = X \cup Y$, where $k = 0$, 1 and

$P_0 = P$. Let $i_\lambda : X \subset P_\lambda$, $j_\lambda : Y \subset P_\lambda$ and let $r_\lambda : P_\lambda \to Y$ denote a retraction ($\lambda = 0, 1, 2$). Then $j_\lambda r_\lambda \simeq 1 : P_\lambda \subset P_\lambda$ and

$$j_2 r_k i_k \simeq j_2 f \simeq j_2 r_2 i_2 \simeq i_2 \quad (k = 0, 1).$$

Therefore it follows from the homotopy extension theorem that $j_2 r_k \simeq g_k : P_k \to P_2$ where g_k maps $X \cup Y$ identically. Let Q_k denote a polyhedral mapping cylinder for g_k [which exists by (8.1)]. Since Q_k retracts onto P_2 which retracts onto Y with $r_2 i_2 \simeq f$, it follows that Q_k is also a polyhedral mapping cylinder for $[f]$. Hence it follows from the preceding paragraph that $\Delta_n(P_k) = \Delta_n(Q_k) = \Delta_n(P_2)$. Therefore

$$\Delta_n(P_0) = \Delta_n(P_1)$$

and $\Delta_n(P)$ depends only on $[f]$.

(*d*) *Properties of the map* $\Delta_n : [X, Y] \to \{D_n Y, D_n X\}$.

We write $\Delta_n(P) = \Delta_n[f]$, thus defining a map

$$\Delta_n : [X, Y] \to \{D_n Y, D_n X\}. \tag{9.7}$$

If $m \geqslant 0$ we have X, $Y \subset S^n \subset S^{n+m}$ and it follows from (9.3), with $q \geqslant n+m$, that

$$\Delta_{n+m} = S^m \Delta_n : [X, Y] \to \{S^m D_n Y, S^m D_n X\}. \tag{9.8}$$

Let $g : Y \to Z$ where Z is a polyhedron in $S^n - (X \cup Y)$. For q sufficiently large let $P \cap Z = \varnothing$ and let $Q \subset S^q$ be a polyhedral mapping cylinder for $[g]$ such that $Q \cap P = Y$. Evidently $P \cup Q$ is a polyhedral mapping cylinder for $[gf]$. Let X^*, P^*, Y^*, etc. be as in (2.2) and (9.1) and consider the diagram

$$\begin{tikzcd}
 & P^* \arrow[d, "\iota_2"'] & Y^* \arrow[l, "\rho_2"'] & Q^* \arrow[l, "\iota_3"'] & \\
D_q X & X^* \arrow[l, "\rho_0"'] & (P \cup Q)^* \arrow[l, "\iota_1"'] \arrow[ul, "\iota_5"] \arrow[ur, "\iota_4"'] & Z^* \arrow[l, "\rho_1"'] \arrow[u, "\rho_3"'] & D_q Z \arrow[l, "\iota_0"']
\end{tikzcd}$$

where ι_λ, ρ_λ^{-1} are inclusions. Arguments similar to those used above show that the diagram is commutative. Therefore

$$\begin{aligned} S^{q-n} \Delta_n[gf] &= \rho_0 \iota_1 \rho_1 \iota_0 = \rho_0 \iota_2 \rho_2 \iota\iota^{-1} \iota_3 \rho_3 \iota_0 \quad (\iota : D_q Y \subset Y^*) \\ &= S^{q-n} \Delta_n[f] S^{q-n} \Delta_n[g], \end{aligned}$$

whence

$$\Delta_n[gf] = \Delta_n[f] \Delta_n[g]. \tag{9.9}$$

B. *Definition of* Δ_n *in general.*

Let $f : X \to Y$, where X, Y denote polyhedra in S^n, with $X \cap Y$ arbitrary, and let $D_n X$, $D_n Y$ denote fixed n-duals of X, Y. For a sufficiently large $q \geqslant n$ let $X_1 \subset S^q - (X \cup Y)$ denote a (polyhedral) copy of X. Let

$h_1 : X \to X_1$ denote a homeomorphism (onto) and define

$$\Delta_n([f], h_1) = S^{n-q}(\Delta_q[h_1]\,\Delta_q[fh_1^{-1}]) \varepsilon \{D_n Y, D_n X\}, \tag{9.10}$$

where Δ_q refers to $D_q W = S^{q-n} D_n W$ for $W = X$, Y. It follows from (9.8) that $\Delta_n([f], h_1)$ does not depend on q. We shall now show that it does not depend on the choice of h_1.

(*e*) *Independence of the choice of* h_1.

Let $X_2 \subset S^q - (X \cup Y)$ also denote a copy of X and let $h_2 : X \to X_2$ denote a homeomorphism. First assume that $X_1 \cap X_2 = \varnothing$. Then it follows from (9.9) that

$$\begin{aligned}\Delta_q[h_2]\,\Delta_q[fh_2^{-1}] &= \Delta_q[h_1]\,\Delta_q[h_2 h_1^{-1}]\,\Delta_q[fh_2^{-1}]\\ &= \Delta_q[h_1]\,\Delta_q[fh_1^{-1}],\end{aligned}$$

whence $\Delta_n([f], h_1) = \Delta_n([f], h_2)$ in this special case.

For the case where $X_1 \cap X_2$ is arbitrary let $X_3 \subset S^q$ denote a copy of X disjoint from $X_1 \cup X_2$ as well as $X \cup Y$ and let $h_3 : X \to X_3$ denote a homeomorphism. Then by the above

$$\Delta_n([f], h_1) = \Delta_n([f], h_3) = \Delta_n([f], h_2),$$

so we may define $\Delta_n[f] = \Delta_n([f], h_1)$. It follows from (9.10) and (9.9) that $\Delta_n[f]$ is the same as in A if $X \cap Y = \varnothing$. Hence we have defined (9.7) for every pair of polyhedra X, $Y \subset S^n$.

(*f*) *Properties of* Δ_n.

It follows from (9.8) and (9.10) that (9.8) is satisfied even if $X \cap Y \neq \varnothing$. Let $f : X \to Y$, $g : Y \to Z$, where X, Y, Z denote polyhedra in S^n. Let $h_1 : X \to X_1$, $k_1 : Y \to Y_1$ denote homeomorphisms where X_1, $Y_1 \subset S^q$ are polyhedra, disjoint from each other and from X, Y, Z. Then it follows from (9.9) for the pairs of maps

$$X_1 \xrightarrow{k_1 f h_1^{-1}} Y_1 \xrightarrow{g k_1^{-1}} Z, \qquad X_1 \xrightarrow{f h_1^{-1}} Y \xrightarrow{k_1} Y_1$$

that

$$\begin{aligned}\Delta_q[h_1]\,\Delta_q[gfh_1^{-1}] &= \Delta_q[h_1]\,\Delta_q[gk_1^{-1} k_1 f h_1^{-1}]\\ &= \Delta_q[h_1]\,\Delta_q[k_1 f h_1^{-1}]\,\Delta_q[gk_1^{-1}]\\ &= \Delta_q[h_1]\,\Delta_q[fh_1^{-1}]\,\Delta_q[k_1]\,\Delta_q[gk_1^{-1}],\end{aligned}$$

so (9.9) holds in general.

Let $i : X \subset Y$ and let $\iota' : D_n Y \subset D_n X$. Let $X \times I = P_1$ be piecewise linearly imbedded in $S^q - (D_q X \cup D_q Y)$ (*e.g.* as part of the cone $X * v_q'$, $q > n$) so that $(x, 0) = x$ for every $x \varepsilon X$ and $P_1 \cap Y = X$. Let $X_1 = X \times 1$ and let $h_1 : X \to X_1$ be defined by $h_1 x = (x, 1)$. Let $P = P_1 \cup Y$. Then P_1, P are mapping cylinders for h_1, ih_1^{-1}. Since $D_q X \subset S^q - P_1$ and X, X_1 are deformation retracts of P_1, we may take $D_q X_1 = D_q P_1 = D_q X$.

Also $D_q Y \subset S^q - P$ and in (9.2), with X replaced by X_1, we may assume $D_q Y \subset P^*$. Since $D_q Y \subset D_q X$ and $D_q X = D_q P_1 = D_q X_1$, it follows from (9.4) that

$$\Delta_q[ih_1^{-1}] : D_q Y \subset D_q X, \quad \Delta_q[h_1] : D_q X \subset D_q X.$$

Therefore $\Delta_q[i] = S^{q-n} \iota' : D_q Y \subset D_q X$, and it follows from (9.8) that

$$\Delta_n[i] = \iota' : D_n Y \subset D_n X. \tag{9.11}$$

In particular, if $i : X \subset X$, then $\Delta_n[i] : D_n X \subset D_n X$. Hence, using (9.9) and (9.11), it follows that, if X is a deformation retract of Y and $r : Y \to X$ is a retraction, then

$$\Delta_n[r] = \iota'^{-1} : D_n X \to D_n Y. \tag{9.12}$$

LEMMA (9.13). *Let X, Y denote polyhedra in S^n and let $f : X \to Y$. If q is sufficiently large, then f is homotopic to a product of inclusion maps and retractions by deformation between polyhedra in S^q.*

Proof: We have $f = (fh_1^{-1})h_1$, where h_1 means the same as in (9.10), and (9.13) follows from (8.1), applied successively to h_1 and fh_1^{-1}.

LEMMA (9.14). *If $f : X \to Y$, where X, Y are polyhedra in S^n, and if $D_{n+1} SW = D_n W$ for $W = X$, Y, then $\Delta_{n+1}[Sf] = \Delta_n[f]$.*

Proof: Let $r > n$ and let S_r denote the suspension operator defined by taking joins with S_r^0, applied to polyhedra in S^{r-1} and maps and S-maps between such polyhedra. Thus $S_r W = W * S_r^0$ and $S_r f = f * e_r$ where $e_r : S_r^0 \subset S_r^0$. If q is sufficiently large it follows from (9.13) that f is homotopic to a product of inclusion maps and retractions by deformation in S^q. If $i : X_1 \subset X_2$, then $S_{q+1} i : S_{q+1} X_1 \subset S_{q+1} X_2$ and if $r : X_3 \to X_4$ is a retraction by deformation, so is $S_{q+1} r$ $(X_\lambda \subset S^q)$. Therefore it follows from (9.11), (9.12), and (9.9) that

$$\Delta_{q+1}[S_{q+1} f] = \Delta_q[f]. \tag{9.15}$$

Let h denote the linear map of R^∞ onto itself which is defined by $hv'_{n+1} = v_{q+1}$, $hv'_{q+1} = v'_{n+1}$, $hv_j' = v_j'$ if $j \neq n+1, q+1$. For every $A \subset S^{q+1}$ let $h_A : A \to hA$ denote the homeomorphism determined by h. Then an isomorphism

$$h_* : \{A, B\} \approx \{hA, hB\} \quad (A, B \subset S^{q+1})$$

is defined by $h_* \alpha = \{h_B\} \alpha \{h_A\}^{-1}$. If $g : A \to B$, let

$$g^h = h_B g h_A^{-1} : hA \to hB.$$

It is easily verified that $\Delta_{q+1}[g^h] = h_* \Delta_{q+1}[g]$; also that $S_{n+1} W = hS_{q+1} W$, $S_{n+1} f = (S_{q+1} f)^h$ and, if

$$S^{q+n} \beta = S_q \cdots S_{n+1} \beta, \quad S_1^{q-n} \beta = S_{q+1} \cdots S_{n+2} \beta,$$

where $\beta \varepsilon \{D_n Y, D_n X\}$, that[5] $h_* S^{q-n}\beta = S_1^{q-n} h_* \beta = S_1^{q-n}\beta$. Hence it follows from (9.8) and (9.15) that

$$\begin{aligned} S_1^{q-n}(\Delta_{n+1}[S_{n+1}f]) &= \Delta_{q+1}[S_{n+1}f] = \Delta_{q+1}[(S_{q+1}f)^h] \\ &= h_* \Delta_{q+1}[S_{q+1}f] = h_* \Delta_q[f] \\ &= h_* S^{q-n}\Delta_n[f] = S_1^{q-n}\Delta_n[f], \end{aligned}$$

whence $\Delta_{n+1}[S_{n+1}f] = \Delta_n[f]$.

C. *Definition and properties of* $D_n : \{X, Y\} \to \{D_n Y, D_n X\}$.

Let $g : S^m X \to S^m Y$ denote a map representing a given $\alpha \varepsilon \{X, Y\}$. It follows from (9.14) that an element $\alpha^* \varepsilon \{D_n Y, D_n X\}$, depending only on α, is defined by $\alpha^* = \Delta_{m+n}[g]$ with $D_{m+n} S^m W = D_n W$ for $W = X, Y$. We define $D_n : \{X, Y\} \to \{D_n Y, D_n X\}$ by $D_n \alpha = \alpha^*$. Then (4.1), (4.2), (4.3) follow from (9.11), (9.9), (9.8) and the definition of $D_n \alpha$. (4.4) is an immediate consequence of (9.13) and the other properties of D_n.

To prove (4.5) note that if $\alpha \varepsilon \{X, Y\}$ or $\{D_n Y, D_n X\}$ it follows from (4.3) that, for $q \geqslant n$, $D_q D_q \alpha = D_q S^{q-n} D_n \alpha = D_n D_n \alpha$. If X_1, Y_1, Z_1 are polyhedra in S^q and if $\alpha_1 \varepsilon \{X_1, Y_1\}$, $\beta_1 \varepsilon \{Y_1, Z_1\}$ and $D_q \alpha_1$, $D_q \beta_1$ refer to the same $D_q Y_1$, then

$$D_q D_q(\beta_1 \alpha_1) = D_q(D_q \alpha_1 D_q \beta_1) = D_q D_q \beta_1 D_q D_q \alpha_1.$$

Hence (4.5) follows from (9.13), (4.1) and the fact that $D_q \iota^{-1} = \iota'^{-1}$ if $\iota : X_1 \subset Y_1$ is an S-equivalence and $\iota' : D_q Y_1 \subset D_q X_1$.

10. *Proof of* (4.10). We observe that if $q \geqslant n$ and

$$D_q W = S^{q-n} D_n W \quad (W = X, Y),$$

then $D_q \alpha = S^{q-n} D_n \alpha$, by (4.3), and the diagram

$$\begin{array}{ccc} H^{n-p-1}(D_n X) & \xrightarrow{(D_n \alpha)^\#} & H^{n-p-1}(D_n Y) \\ \downarrow{\scriptstyle S^{q-n}} & & \downarrow{\scriptstyle S^{q-n}} \\ H^{q-p-1}(D_q X) & \xrightarrow{(D_q \alpha)^\#} & H^{q-p-1}(D_q Y) \end{array}$$

is commutative. Moreover ([5]), $S^{q-n} \mathfrak{D}_n u = \mathfrak{D}_q u$ for $u \varepsilon H_q(W)$. It follows that the integer n in (4.9) may be replaced by an arbitrary $q > n$. By (9.13) we see that it suffices to prove (4.10) in the case $\alpha : X \subset Y$. If (4.10) is true for one pair of n-duals $D_n X$, $D_n Y$, it is obviously true for any other. Therefore (4.10) follows from the commutativity of (2.1).

[5] Since the operator S_i is defined by a geometrical construction in R^∞ we have $S_i S_j = S_j S_i$. (N.B.—Except for the ordering of v_i, v_i' there are no orientations to be considered.)

11. *Proof of* (4.13). Let P, P' denote disjoint polyhedra in S^q and $A_1, \ldots, A_m$ polyhedra whose union is S^q. For every non-empty subset, τ, of $(1, \ldots, m)$ let $A_\tau = \bigcap_{i \varepsilon \tau} A_i$.

LEMMA (11.1). *If $P' \cap A_\tau$ is an S-deformation retract of $A_\tau - P$ for each τ, then P' is q-dual to P.*

Proof. The lemma is trivial if $m = 1$. If $m > 1$ let $B = A_1 \cup A_2$ and let C be either S^q or, if $m > 2$, A_τ, where τ denotes a non-empty subset of $(3, \ldots, m)$. Then we have triads $(P' \cap B \cap C;\ P' \cap A_1 \cap C,\ P' \cap A_2 \cap C)$ and $\big((B \cap C) - P;\ (A_1 \cap C) - P,\ (A_2 \cap C) - P\big)$ each of which can be triangulated to form a CW-complex and a pair of subcomplexes. Therefore, it follows from the hypothesis of (11.1) and the 5-lemma ([**6**], page 16), applied to the Mayer-Vietoris sequences[6] ([**6**], page 39) of these triads, that

$$i_{\#} : H_k(P' \cap B \cap C) \approx H_k\big((B \cap C) - P\big) \quad (k = 0, 1, \ldots),$$

where $i_{\#}$ is the injection. Therefore $P' \cap B \cap C$ is an S-deformation retract of $(B \cap C) - P$ and $A_1, \ldots, A_m$ may be replaced by $B, A_3, \ldots, A_m$. The lemma now follows by induction on m.

By a *polyhedral n-element* (in R^∞) we mean a piecewise linear homeomorph of I^n. Let K_n be the standard triangulation of S^n defined in §2, let $\sigma_0{}^n$ be a simplex of K_n and let σ^n be a rectilinear simplex in $\sigma_0{}^n - \dot{\sigma}_0{}^n$.

LEMMA (11.2). *$S^n - (\sigma^n - \dot{\sigma}^n)$ is a polyhedral n-element*[7].

Proof. Let $E_0{}^n = S^n - (\sigma_0{}^n - \dot{\sigma}_0{}^n)$, $P^n = \sigma_0{}^n - (\sigma^n - \dot{\sigma}^n)$ and let

$$x = (0, 0, \ldots) \varepsilon R^\infty.$$

On projecting from an inner point of σ^n it follows from Theorem 5 in [**12**] that $E_0{}^n$, P^n are piecewise linearly homeomorphic to $x * \dot{\sigma}_0{}^n$, $\dot{E}_0{}^n \times I$ respectively. (*N.B.*: $\dot{\sigma}_0{}^n = \dot{E}_0{}^n$.) Therefore $E_0{}^n$, likewise $E_0{}^n \cup P^n$, are polyhedral n-elements. But $E_0{}^n \cup P^n = S^n - (\sigma^n - \dot{\sigma}^n)$ and (11.2) is proved.

As obvious, and well-known, corollaries of (11.2) we have:

COROLLARY (11.3). *Let $E_1{}^n$, $E_2{}^n \subset S^n$ be polyhedral n-elements such that $E_1{}^n \cup E_2{}^n = S^n$ and $E_1{}^n \cap E_2{}^n = \dot{E}_1{}^n = \dot{E}_2{}^n$. Then there is a piecewise linear homeomorphism $h : S^n \to S^n$ such that $h\sigma^n = E_1{}^n$.*

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[6] Let A denote a subcomplex of a CW-complex X. Then $H_r(X, A)$ may be calculated combinatorially in terms of the cells in $X - A$. Thus the homology groups of (X, A) may be regarded as the homology groups of the " open subcomplex " $X - A$. Therefore the strong excision theorem ([**6**], p. 165) is valid in the category of CW-complexes and subcomplexes.

[7] This is a special case of Theorem [14 . 2] in [**1**] (*cf.* Theorem 5 in [**12**] and §15 in [**1**]). We indicate an *ad hoc* proof, leaving some details to be supplied by the reader.

COROLLARY (11.4). *Let a, b and a', b' be pairs of distinct points in S^{n-1}. Then there is a piecewise linear homeomorphism $h: S^{n-1} \to S^{n-1}$ such that $ha = a'$, $hb = b'$.*

We now prove (4.13). It is trivial if $n = 0$ so we assume that $n \geqslant 1$. Let $E_1^n = S^{n-1} * v_n$, $E_2^n = S^{n-1} * v_n'$ and let a, a^* be distinct points in S^{n-1}. We proceed to prove that there is a piecewise linear homeomorphism $f_\lambda: S^n \to S^n$ $(\lambda = 1, 2)$ such that

$$f_\lambda(P_\lambda \cup P_\lambda^*) \subset E_\lambda^n, \ S^{n-1} \cap f_\lambda P_\lambda = a, \ S^{n-1} \cap f_\lambda P_\lambda^* = a^*. \quad (11.5)$$

Since $P_\lambda \cup P_\lambda^*$ is a closed, proper subset of S^n there is a simplex σ^n, which is interior to some simplex of K_n and does not meet $P_\lambda \cup P_\lambda^*$. Let $E^n = S^n - (\sigma^n - \dot{\sigma}^n)$. Then it follows from (11.2), (11.3) that there is a piecewise linear homeomorphism $S^n \to S^n$ which interchanges E^n and σ^n. So we assume from the outset that $P_\lambda \cup P_\lambda^* \subset \sigma^n - \dot{\sigma}^n$. Let $R_0^n \subset R^\infty$ be the linear n-space which contains σ^n and let δ be the Euclidean distance function which R_0^n derives from Hilbert space. Let $p_\lambda \varepsilon P_\lambda$, $p_\lambda^* \varepsilon P_\lambda^*$ be points such that $\delta(p_\lambda, p_\lambda^*) = \delta(P_\lambda, P_\lambda^*)$ and let V^n be the interior and boundary of the metric n-sphere in R_0^n which has $p_\lambda p_\lambda^*$ as a diameter. Then $V^n \cap (P_\lambda \cup P_\lambda^*) = p_\lambda \cup p_\lambda^*$ and $V^n \cap \sigma^n$ is a convex subset of σ^n. Hence there is obviously an n-simplex $\sigma_1^n \subset V^n \cap \sigma^n$ which has p_λ, p_λ^* among its vertices. Therefore it follows from (11.3), (11.4) that there is a piecewise linear homeomorphism $f_\lambda: S^n \to S^n$ such that

$$f_\lambda \sigma_1^n = E_\mu^n \ (\lambda \neq \mu = 1 \text{ or } 2), \ f_\lambda p_\lambda = a, \ f_\lambda p_\lambda^* = a^*.$$

Clearly f_λ satisfies (11.5).

We take p_λ, p_λ^* to be the base points for $P_1 \vee P_2$, $P_1^* \vee P_2^*$. Then piecewise linear homeomorphisms h, h', of $P_1 \vee P_2$, $P_1^* \vee P_2^*$ into S^n, are defined by

$$h(p, p_2) = f_1 p, \qquad h(p_1, q) = f_2 q \qquad (p \varepsilon P_1, \ q \varepsilon P_2),$$

$$h'(p^*, p_2^*) = f_1 p^*, \quad h'(p_1^*, q^*) = f_2 q^* \quad (p^* \varepsilon P_1^*, \ q^* \varepsilon P_2^*).$$

Let $X_\lambda = f_\lambda P_\lambda$, $X_\lambda^* = f_\lambda P_\lambda^*$. Since $P_\lambda^* = D_n P_\lambda$ it is an S-deformation retract of $S^n - P_\lambda$. So therefore is X_λ^* of $S^n - X_\lambda$, whence X_λ^* is n-dual to X_λ. Evidently $S^{n-1} - a$ is a deformation retract of $E_\mu^n - a$, whence $E_\lambda^n - X_\lambda$ is a deformation retract of $S^n - X_\lambda$. Therefore X_λ^* is an S-deformation retract of $E_\lambda^n - X_\lambda$. Also a^* is an S-deformation retract of $S^{n-1} - a$ and it follows from (11.1) that $X_1^* \cup X_2^*$ is n-dual to $X_1 \cup X_2$.

Let K_μ be a triangulation of E_μ^n, which has a for a vertex, and let $A_\mu \subset K_\mu$ be the subcomplex complementary to a. Let the mesh of K_μ be so small that $X_\mu^* \subset A_\mu$. Since $E_\mu^n - a$ is contractible, so is A_μ. We have $A_\mu \subset S^n - X_\lambda$, $A_\mu \cap X_\lambda^* = a^*$. Hence it follows that X_μ^* is a deformation retract of $A_\mu \cup X_\lambda^*$ and that $A_\mu \cup X_\lambda^*$ is n-dual to X_λ. Therefore $D_n \iota_\lambda = \rho_\lambda'' \iota_\lambda'' = \rho_\lambda'$, where

$$X_1^* \cup X_2^* \xrightarrow{\iota_\lambda''} A_\mu \cup X_\lambda^* \xrightarrow{\rho_\lambda''} X_\lambda^*$$

are the inclusion and the S-retraction by deformation. Similarly $D_n \iota_\lambda' = \rho_\lambda$ and (4.13) is proved.

12. *Proof that D_n is a homomorphism.* Since $S: \{X, Y\} \approx \{SX, SY\}$ and $D_n = D_{n+1} S$, by (4.3), we may replace D_n by

$$D_{n+1}: \{SX, SY\} \to \{D_n Y, D_n X\}.$$

Therefore we may assume to begin with that $X \neq \varnothing$ and, since D_n may also be replaced by SD_n, that $X \neq S^n$. Let $X \neq \varnothing$ or S^n and let $X_1 \cup X_2$, $X_1^* \cup X_2^*$, ι_λ, ρ_λ, ι_λ', ρ_λ' be as in (4.13) with

$$P_1 = P_2 = X, \quad P_1^* = P_2^* = D_n X.$$

Let $h_\lambda: X \to X_\lambda$ be a homeomorphism onto X_λ, let $\beta_\lambda = \{h_\lambda\}$ and let

$$\beta = \iota_1 \beta_1 + \iota_2 \beta_2 : X \to X_1 \cup X_2.$$

On considering the track addition of maps $S(X_1 \cup X_2) \to S(X_1 \cup X_2)$ it follows without difficulty that

$$\iota_1 \rho_1 + \iota_2 \rho_2 : X_1 \cup X_2 \subset X_1 \cup X_2.$$

Similarly, and from (4.13), it follows that

$$D_n(\iota_1 \rho_1) + D_n(\iota_2 \rho_2) : X_1^* \cup X_2^* \subset X_1^* \cup X_2^*. \tag{12.1}$$

Let $\alpha_1, \alpha_2 \in \{X, Y\}$ be given and let $f_\lambda: S^k X \to S^k Y$ $(k \geqslant 1)$ be a map representing α_λ such that $f_\lambda S^k x_\lambda = v_{n+k}$, where $x_\lambda = h_\lambda^{-1}(X_1 \cap X_2)$. Define $g: S^k(X_1 \cup X_2) \to S^k Y$ by $gx = f_\lambda(S^k h_\lambda^{-1})x$ if $x \in X_\lambda$ and let $\gamma: X_1 \cup X_2 \to Y$ be the S-map represented by g. Then $\gamma \iota_\lambda = \alpha_\lambda \beta_\lambda^{-1}$, whence $\gamma \iota_\lambda \beta_\lambda = \alpha_\lambda$. Therefore

$$\alpha_1 + \alpha_2 = \gamma(\iota_1 \beta_1 + \iota_2 \beta_2) = \gamma \beta. \tag{12.2}$$

Obviously $\rho_\lambda \iota_\lambda : X_\lambda \subset X_\lambda$ and $\rho_\lambda \iota_\mu = 0$ if $\mu \neq \lambda$. Therefore $\beta_\lambda = \rho_\lambda \beta$, whence $\gamma \iota_\lambda \rho_\lambda \beta = \alpha_\lambda$. Hence it follows from (12.2), (12.1) that

$$\begin{aligned} D_n(\alpha_1 + \alpha_2) &= D_n(\beta) \left(D_n(\iota_1 \rho_1) + D_n(\iota_2 \rho_2) \right) D_n(\gamma) \\ &= D_n(\gamma \iota_1 \rho_1 \beta) + D_n(\gamma \iota_2 \rho_2 \beta) \\ &= D_n \alpha_1 + D_n \alpha_2 \end{aligned}$$

and the proof is complete.

13. *Proof of* (6.2). Let A, B, A', B' denote CW-complexes, let $f: A \to B$, $f': A' \to B'$ be cellular maps and in the diagram

$$\begin{array}{ccc} A & \xrightarrow{\{f\}} & B \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ A' & \xrightarrow{\{f'\}} & B' \end{array}$$

let α, β denote S-maps such that $\beta\{f\} = \{f'\}\alpha$. Let Z_f, $Z_{f'}$ be defined as in §6, likewise $g: Z_f \to SA$, $g': Z_{f'} \to SA'$.

LEMMA (13.1). *There is an S-map $\zeta: Z_f \to Z_{f'}$ such that the diagram*

$$\begin{array}{ccccc} B & \xrightarrow{\{i\}} & Z_f & \xrightarrow{\{g\}} & SA \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\zeta} & & \downarrow{\scriptstyle S\alpha} \\ B' & \xrightarrow{\{i'\}} & Z_{f'} & \xrightarrow{\{g'\}} & SA' \end{array}$$

is commutative, where $i: B \subset Z_f$, $i': B' \subset Z_{f'}$. Furthermore, if α, β are S-equivalences, so is ζ.

Proof: It is easily verified that $SZ_f = Z_{Sf}$. Therefore we may replace the above diagrams by their k-fold suspensions for any $k \geqslant 0$. Hence we may assume that $\alpha = \{u\}$, $\beta = \{v\}$, where $u: A \to A'$, $v: B \to B'$ are maps such that $vf \simeq f'u$. This being so, let $h_t: A \to B'$ be a homotopy such that $h_0 = vf$, $h_1 = f'u$. Then a map $w: Z_f \to Z_{f'}$ is defined by $wb = vb$ if $b \varepsilon B$ (whence $wi = i'v$) and

$$w(a, t) = \begin{cases} h_{2t}a & \text{if } 0 \leqslant t \leqslant \frac{1}{2} \\ (ua, 2t-1) & \text{if } \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

for $a \varepsilon A$. Let p, p' denote the (ordered) poles of SA'. Then $(Su)gB = g'wB = p$ and

$$(Su)\,g(a, t) = Su(a, 2t-1) = (ua, 2t-1),$$

$$g'w(a, t) = \begin{cases} p & \text{if } 0 \leqslant t \leqslant \frac{1}{2} \\ (ua, 4t-3) & \text{if } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Hence, obviously, $(Su)g \simeq g'w$ and the first part of the lemma follows on taking $\zeta = \{w\}$.

To prove the second part consider the sequence

$$\cdots \longrightarrow H_q(A) \xrightarrow{f_\#} H_q(B) \xrightarrow{i_\#} H_q(Z_f) \xrightarrow{S^{-1}g_\#} H_{q-1}(A) \longrightarrow \cdots,$$

where $f_\#$, $i_\#$, $g_\#$ are the homomorphisms induced by f, i, g. It follows without difficulty from the excision theorem that this sequence is isomorphic to the homology sequence of the pair (Z_f, B). Therefore it is exact. Also it is natural with respect to $\alpha_{\#}$, $\beta_{\#}$, $\zeta_{\#}$. Hence, if $\alpha_{\#}$, $\beta_{\#}$ are isomorphisms onto for every q, so is $\zeta_{\#}$, by the 5-lemma. Therefore, if α, β are S-equivalences, so is ζ and the proof is complete.

We now prove (6.2). In (6.1) let $\alpha = \{f\}$, $\alpha^* = \{f^*\}$. Then we have to prove that there are S-equivalences ζ, ζ^* such that the diagrams

$$\begin{tikzcd} Y \arrow[r, "\{i\}"] \arrow[d, "\eta"] & Z \arrow[r, "\{g\}"] \arrow[d, "\zeta"] & SX \arrow[d, "S\xi,"] \\ Y_0 \arrow[r, "\lambda"] & Z_0 \arrow[r, "\mu"] & SX_0 \end{tikzcd} \qquad \begin{tikzcd} SY^* & Z^* \arrow[l, "\{g'\}"'] & X^* \arrow[l, "\{i'\}"'] \\ SY_0^* \arrow[u, "S\eta^*"] & Z_0^* \arrow[l, "-\lambda^*"'] \arrow[u, "\zeta^*"] & X_0^* \arrow[l, "\mu^*"'] \arrow[u, "\xi^*"] \end{tikzcd} \tag{13.2}$$

are commutative, where $X_0^* = D_n X_0$, $Y_0^* = D_n Y_0$, Z_0, Z_0^* are mutually $(n+1)$-dual polyhedra in S^{n+1} and $\lambda^* = D_{n+1}\lambda$, $\mu^* = D_{n+1}\mu$. Let $\xi_1 : X \to X_1$, $\eta_1 : Y \to Y_1$ be S-equivalences, where X_1, Y_1 are polyhedra in S^n, and let

$$\theta = \xi_1 \xi^{-1} : X_0 \to X_1, \ \phi = \eta_1 \eta^{-1} : X_0 \to Y_1.$$

Let $D_n W_1$ be any n-dual of W_1 ($W = X$, Y. Possibly $W_1 = W_0$, $D_n W_1 \neq D_n W_0$). Let (6.1)$'$, (13.2)$'$ denote (6.1), (13.2) with ξ, η, ξ^*, η^*, λ, μ replaced by ξ_1, η_1, $\xi^* D_n \theta$, $\eta^* D_n \phi$, $\lambda\phi^{-1}$, $(S\theta)\mu$. Then it follows from (4.2) that (6.1), (6.1)$'$ define the same $D_n\{f\}$ and that (13.2)$'$ are commutative if, and only if, (13.2) are commutative. Therefore we may choose ξ, η and the n-duals $D_n X_0$, $D_n Y_0$ to suit our convenience.

Since $n \geqslant 2 \dim Z + 1$ there are disjoint polyhedra X_0, $Y' \subset S^n$ of the same dimensionalities and homotopy types as X, Y. Let

$$u : X \to X_0, \ v : Y \to Y'$$

be homotopy equivalences, let $u' : X_0 \to X$ be a homotopy inverse of u and let $f_0 = vfu' : X_0 \to Y'$. Let $Y_0 \subset S^n$ be a polyhedral mapping cylinder for f_0 (*N.B.*: $\dim Y_0 = \dim Z$) and let $\xi = \{u\}$, $\eta = \{lv\}$, where $l : Y' \subset Y_0$. Then

$$\eta\{f\}\xi^{-1} = \{lvfu'\} = \{lf_0\} : X_0 \subset Y_0.$$

Thus $\eta\{f\}\xi^{-1} = \{j\}$, where $j : X_0 \subset Y_0$. Let

$$Z_0 = Z_j = Y_0 \cup (X_0 * v'_{n+1}).$$

Let $t_{n+1}(p)$ denote the $(n+1)$th coordinate of a given point $p \in R^\infty$. We represent points in S^{n+1} by (a, s), where $a = (a, 0) \in S^n$ and $s = 2t_{n+1}(a, s)$ $\big((a, s) \in a * S^0_{n+1}\big)$. Thus $(a, -2) = v_{n+1}$, $(a, 2) = v'_{n+1}$ and if $A \subset S^n$ we imbed $A \times I$ in S^{n+1} so that $(a, t) \in S^{n+1}$ has its usual meaning in $A \times I$ ($a \in A$, $t \in I$). We write $A \times I = A_I$, $A \times 1 = A_1$ and

$$(A * v_{n+1}) \cup A_I = A_1 \# v_{n+1}.$$

Let $X_0^* \subset S^n$ and $Y_0^* \subset X_0^*$ be n-duals of X_0, Y_0, let $W_1^* = (W_0^*)_1$ and let $h : X_1^* \to X_0^*$ be defined by $h(x, 1) = x$ ($x \in X_0^*$). Let $j^* : Y_0^* \subset X_0^*$,

$j' = h^{-1}j^* : Y_0^* \to X_1^*$ and let

$$Z_0^* = Z_{j'} = X_1^* \cup (Y_1^* \# v_{n+1}).$$

Clearly $X_0^* \times I \subset S^{n+1} - SX_0$ and since X_0^* is $(n+1)$-dual to SX_0 so is $X^* \times I$ and hence also X_1^*. Moreover $\{h\}$ is $(n+1)$-dual to $\iota : SX_0 \subset SX_0$. Since the right-hand diagram in (6.1), with $\alpha_0 = \{j\}$, $D_n \alpha_0 = \{j^*\}$, $\alpha^* = \{f^*\}$, is commutative, so is

$$\begin{array}{ccc} X^* & \xleftarrow{\{f^*\}} & Y^* \\ {\scriptstyle \xi^*\{h\}}\uparrow & & \uparrow{\scriptstyle \eta^*} \\ X_1^* & \xleftarrow{\{j'\}} & Y_0^* \end{array} \qquad (13.3)$$

In (11.1) let $P = Z_0$, $P' = Z_0^*$ and let $A_1, \ldots, A_m$ denote

$$S_1^n * v'_{n+1}, \; S_I^n, \; S^n * v_{n+1} \quad (q = n+1, \; m = 3).$$

Then a homotopy $h_t : S_I^n - Z_0 \to S_I^n - Z_0$, rel. $S_1^n - X_1$, such that

$$h_0 = 1, \; h_t(Z_0^* \cap S_I^n) \subset Z_0^* \cap S_I^n$$

and $$h_1(S_I^n - Z_0) = S_1^n - X_1, \quad h_1(Z_0^* \cap S_I^n) = X_1^*,$$

is defined by $h_t(a, s) = \big(a, (1-t)s + t\big)$, for $(a, s) \in S_I^n - Z_0$. Since X_0^* is an S-deformation retract of $S^n - X_0$, so is X_1^* of $S_1^n - X_1$ and it follows that $Z_0^* \cap S_I^n$ is an S-deformation retract of $S_I^n - Z_0$. It is obvious that $Z_0^* \cap A_\tau$ is an S-deformation retract of $A_\tau - Z_0$ for the remaining sets τ. Therefore it follows from (11.1) that Z_0^* is $(n+1)$-dual to Z_0.

We have

$$Y_0 \xrightarrow{i_0} Z_0 \xrightarrow{g_0} SX_0, \qquad SY_0^* \xleftarrow{g_0'} Z_0^* \xleftarrow{i_0'} X_1^*,$$

where i_0, i_0' are inclusions and g_0, g_0' are defined in the same way as g, g'. In (13.2) let $\lambda = \{i_0\}$, $\mu = \{g_0\}$.

Let $D'_{n+1} Y_0 = (Y_1^* \# v_{n+1}) \cup (S_1^n * v'_{n+1})$. Then $D'_{n+1} Y_0$ and SY_0^* are both $(n+1)$-dual to Y_0. Let $i_1 : Z_0^* \subset D'_{n+1} Y_0$ and let $r : D'_{n+1} Y_0 \to SY_0^*$ be defined by

$$r(a, s) = \begin{cases} \big(a, (4s+2)/3\big) & \textit{if } -2 \leqslant s \leqslant 1, \\ v'_{n+1} & \textit{if } 1 \leqslant s \leqslant 2. \end{cases}$$

Then $\lambda^* = D_{n+1}\{i_0\} = \{ri_1\}$. But $ri_1 = wg_0'$, where $w : SY_0^* \to SY_0^*$ is the "reflexion" which interchanges $y * v_{n+1}$, $y * v'_{n+1}$, with $wy = y$, for every $y \in Y_0^*$. Clearly $\{wg_0'\} = -\{g_0'\}$, whence $-\lambda^* = \{g_0'\}$.

Similarly $D''_{n+1}\{i_0'\} = \{g_0\}$ (not $-\{g_0\}$ because the "vertex" of Z_0 is v'_{n+1} and the analogue of r maps Y_0 on v_{n+1}), where D''_{n+1} refers to $D''_{n+1} Z_0 = Z_0^*$, $D''_{n+1} SX_0 = X_1^*$.

Let $(13.2)_1$ denote (13.2) with X_0^*, ξ^*, μ^* replaced by X_1^*, $\xi^*\{h\}$, $\mu^*\{h\}$, where $\{h\}$ means the same as in (13.3). Since $\{h\}$ is $(n+1)$-dual to

$\iota: SX_0 \subset SX_0$ we have $\mu^*\{h\} = D''_{n+1}\{g_0\} = \{i_0'\}$. Therefore, and since $-\lambda^* = \{g_0'\}$, $\{j\} = \eta\{f\}\xi^{-1}$ and (13.3) is commutative, it follows from (13.1) that there are S-equivalences ζ, ζ^* such that the diagrams $(13.2)_1$ are commutative. Hence, obviously, (13.2) are commutative and (6.2) is proved.

Notice that, if the parts played by v_{n+1}, v'_{n+1} in the above argument are interchanged, so that Z_0, $Z_0{}^*$ are reflected through S^n, then we are led to a weak duality between $\{i\}$, $\{g\}$ and $\{g'\}$, $-\{i'\}$.

References.

1. J. W. Alexander, "The combinatorial theory of complexes", *Annals of Math.*, 31 (1930), 292–320.

2. P. Alexandroff and H. Hopf., *Topologie*, Berlin (1935).

3. M. G. Barratt, "Track groups (1)", *Proc. London Math. Soc.* (3), 5 (1955), 71–106.

4. S. C. Chang, "Homotopy invariants and continuous mappings", *Proc. Royal Soc.*, A, 202 (1950), 253–63.

5. B. Clarke, "A note on Alexander's duality" (to appear).

6. S. Eilenberg and N. E. Steenrod, *Foundations of Algebraic Topology*, Princeton (1952).

7. W. S. Massey, "Exact couples in algebraic topology", *Annals of Math.*, 56 (1952), 353–96; 57 (1953), 248–56.

8. M. Nakaoka, "Exact sequences $\Sigma_p(K, L)$ and their applications", *J. Inst. Polytech.*, Osaka City Univ., 3 (1953), 83–100.

9. J.-P. Serre, "Homologie singulière des espaces fibrés", *Annals of Math.*, 54 (1951), 425–505.

10. E. H. Spanier, "Borsuk's cohomotopy groups", *Annals of Math.*, 50 (1949), 203–45.

11. E. H. Spanier and J. H. C. Whitehead, "The theory of carriers and S-theory", *Annals of Math.* (to appear).

12. J. H. C. Whitehead, "On subdivisions of complexes", *Proc. Camb. Phil. Soc.*, 31 (1935), 69–75.

13. ———, "On the groups $\pi_r(V_{n,m})$ and sphere bundles", *Proc. London Math. Soc.* (2), 48 (1944), 243–91.

14. ———, "On simply connected 4-dimensional polyhedra", *Comm. Math. Helvetici*, 22 (1949), 48–92.

15. ———, "Combinatorial homotopy I", *Bull. Amer. Math. Soc.*, 55 (1949), 213–45.

16. ———, "On the realizability of homotopy groups", *Annals of Math.*, 50 (1949), 261–3.

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