Introduction

I supervised a reading group on Folland's "Real analysis" the fall of 2023. I assigned the weekly reading and problem sets. I am an Operator algebraist so I did focus on functional analytic ideas in the latter half.

This document is just a master document of all the problem sets and readings I assigned.

Week 1

We read ch 1.1-1.3.

Hw was exercise 7-12 for the first week.

The following was a bonus problem: We saw the Borel sigma algebra is a natural sigma algebra on a topological space. There is another one, can you guess what it should be?

Week 2

We read section 2.1-2.3 for this week.

Problem 1

We will learn an important technique here, that you can "approximate" measurable sets by certain nice sets.

a. (Hard) Let $(X, \mathcal{M}, \mu)$ be a finite measure space, and let $\mathcal{A} \subset \mathcal{M}$ be an algebra of sets such that $\mathcal{A}$ generates $\mathcal{M}$ as a σ-algebra. Then $\mathcal{A}$ is dense in the measure algebra (see problem 12 from the last PSET), i.e for every $M \in \mathcal{M}$, and $\varepsilon > 0$ there is a $A \in \mathcal{A}$ such that $\mu(M \Delta A) < \varepsilon$.

b. How well can you extend this to σ-finite spaces?

c. Prove proposition 1.20, that for every Lebesgue measurable set of finite measure $E \subset \mathbb{R}$ and $\varepsilon > 0$ there is some open set U such that $m(E \Delta U) < \varepsilon$.

Lets see an application of this idea now:

d. (Problem 30 in the book) Let $E \subset \mathbb{R}$ be lebesgue measurable with positive measure. For each $\alpha < 1$ there is an open interval I such that $m(E \cap I) > \alpha m(I)$.

Problem 2

We say a measure space $(X, \mathcal{M}, \mu)$ is complete if for each measurable set N with $\mu(N) = 0$, all of its subsets are also measurable.

a. Let μ be a measure induced by an outer measure μ^* on X. Let $\mathcal{M}$ be the σ-algebra of μ^* measurable sets. Then $(X, \mathcal{M}, \mu)$ is complete.

b. Show that $(\mathbb{R}, \mathcal{B}, \mu)$, i.e real numbers with Borel σ-algebra and Lebesgue measure, is **not** complete.

c. Show that $(\mathbb{R}, \mathcal{L}, m)$, Lebesgue measurable sets with measure induced by Lebesgue outer measure, is actually the smallest complete measure space containing the space from the previous problem.

This combined tells us that in this sense, if we replace the Borel σ-algebra with the one of Lebesgue measurable set, it is a compeletion.

Problem 3

a. Let $(X_i, \mathcal{M}_i, \mu_i)$ be a finite collection of measure spaces, define

$$X = \prod_i X_i \, , \, \mathcal{M} = \bigotimes_i \mathcal{M}_i.$$

Here $\mathcal{M}$ is the sigma algebra generated by sets of the form $\prod E_i, E_i \in \mathcal{M}_i$. Extend the μ_i to a measure on $(X, \mathcal{M})$.

b. Write down a version of this when the collection is infinite, where the sigma algebra will be generated by **Cylinder Sets**, i.e sets of the form:

$$\prod_i E_i \qquad E_i \in \mathcal{M}_i \text{ and } E_i = X_i \text{ for all but finitely many } i.$$

What modifications do you have to make to the previous part to make this work?

Week 3

Finished chapter 2 for this week.

Problem 1

These are the section exercises I want you to do:

1. Prove proposition 2.20, that if f is a positive integrable function on a measure space, then it is infinite on a null set, and that it is non-zero on a σ-finite set.

2. Prove the generalized DCT in exercise 20.

3. Do exercise 27.

4. Do the computations in exercise 28.

5. Do exercise 29.

Problem 2

For the following problem, let $(X, \mathcal{M}, \mu)$ be measure space.

a. Let $f_n \in L^1(X)$ with $f_n \to f$ in L^1, show that a subsequence of this converges to f a.e. (Bonus: exhibit a counter example to show the whole sequence doesnt need to converge a.e to f.)

b. If (f_n) is a cauchy sequence in L^1, show that it has a subsequence that converges to some f a.e.

c. Let the setting be as in part b. Show that the subsequence can be chosen so that f is in L^1. (Hint=)

d. Now finally show that $L^1(X)$ is complete.

Problem 3

Let $(X, \mathcal{B}_X, \mu)$ be a measure space, where X is a locally compact Haussdorf space, $\mathcal{B}_X$ its Borel algebra, and μ a regular measure that is finite on compact sets. Let $C_c(X)$ be the compactly supported continuous functions $X \longrightarrow \mathbb{C}$, show that $C_c(X) \subset L^1(X)$ and that it is infact a dense subset.

Week 4

Did section 3.1-3.3

section exercises

Do 33, 34, 40, 44 and 52.

Problem 1

Let (X, M_X, μ) be a measure space and (Y, M_Y) a measurable space. Let $T : X \longrightarrow Y$ be measurable.

a. Show ν which is defined as $\nu(A) := \mu(T^{-1}(A))$ is a measure on Y.

b. Show that $f \in L^1(Y, \nu) \iff f \circ T \in L^1(X, \mu)$ and that in this case,

$$\int_X f \circ T \, \mathrm{d}\mu = \int_Y f \, \mathrm{d}\nu$$

Week 5

Problem 1

This was 3 weeks worth of homework. Assigned reading was chapter 5 and 6.

A Banach space is a real (or complex) vector space $\mathfrak{X}$, with a norm $||-||$ such that the space is complete under the induced metric of the norm. We already saw that L^1 was a Banach space.

a. Show that a linear operator $T : \mathfrak{X} \longrightarrow \mathfrak{Y}$ is continuous iff there is some c such that $||Tx||_{\mathfrak{Y}} \leq c||x||_{\mathfrak{X}}$. We call these operators bounded also.

b. For a real Banach space $\mathfrak{X}$, we call the space of bounded linear functionals $\mathfrak{X}^*$. I.e

$$\mathfrak{X}^* := \{L \,|\, L : \mathfrak{X} \longrightarrow \mathbb{R}, L \text{ bounded and linear}\}.$$

What is a good definition for the operator norm $||-||_{op}$ on $\mathfrak{X}^*$ based off of a? Show that it is a Banach space under this norm.

Problem 2

Let $(X, \mathcal{M})$ be a measurable space. Let $M(X)$ be the space of all complex measures on X, with the norm $|| - ||_{TV}$, where $||\lambda|| = |\lambda|(X)$ and $|\lambda|$ is the total variation of λ.

a. Show that $M(X)$ is a complex Banach space.

b. Fix a positive measure μ on X. Show that $L^1(\mu) \longrightarrow M(X)$ that sends f to $f\,\mathrm{d}\mu$ is a linear isometric embedding. (Isometric means norm is preserved)

Optional problems to do after finishing the course:

c. Let μ and ν be mutually singular, then exhibit an isometric embedding

$$L^1(\mu) \oplus^{\ell^1} L^1(\nu) \longrightarrow M(X)$$

where $L^1(\mu) \oplus^{\ell^1} L^1(\nu)$ is the completion of the algebraic direct sum under the norm $\|(f, g)\| = \|f\|_1 + \|g\|_1$.

d. If (X, Σ, μ) is decomposible, i.e there are finite measurable sets X_i with $X = \coprod X_i$, then

$$L^1(\mu) \cong \ell^1 - \bigoplus_i L^1(\mu|_{X_i}).$$

e. Show that $M(X)$ is actually isometrically isomorphic to some $L^1(Y)$.

Problem 3

Let $(X, \mathcal{M}, \mu)$ be a measure space. Define $||f||_p := \left(\int_X |f|^p \,\mathrm{d}\mu\right)^{1/p}$. Then $L^p(X, \mathbb{R})$ is the Banach space of measurable functions $f : X \longrightarrow \mathbb{R}$ such that the p-norm $||f||_p$ is finite, and we consider 2 functions to be equal if they are equal a.e. The proof that this is Banach is the same as that of L^1.

Throughout this problem, $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

a. Prove Young's Inequality, that for positive a, b

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

b. Prove Hölder's Inequality, that if $f \in L^p(X, \mathbb{R})$, $g \in L^q(X, \mathbb{R})$, then

$$\int_X fg \, \mathrm{d}\mu \leq \left(\int_X |f|^p \, \mathrm{d}\mu \right)^{1/p} \left(\int_X |g|^q \, \mathrm{d}\mu \right)^{1/q}.$$

In other words $||fg||_1 \leq ||f||_p ||g||_q$ and that $fg \in L^1(X, \mathbb{R})$. (Bonus: use this to prove Minkowsi's inequality, $||f + g||_p \leq ||f||_p + ||g||_p$, i.e that the p-norm is actually a norm)

c. Define

$$T : L^q(X, \mathbb{R}) \longrightarrow (L^p(X, \mathbb{R}))^*, \quad (T(f))(g) = \int_X fg \, \mathrm{d}\mu.$$

Show that this is well defined, and that it is an isometric embedding.

Now we will show the hard part, that the map above is actually also surjective. This is only true for σ-finite spaces, so we will be assuming this from now on.

d. Let $\Lambda \in (L^p(X, \mathbb{R}))^*$ be positive (i.e sends positive functions to positive numbers), define

$$\lambda(A) := \sup\{\Lambda(\chi_E) | E \subset A, \mu(E) < \infty\}.$$

Show that λ is a measure, $L^p(\mu) \subset L^1(\lambda)$ and that $\Lambda(f) = \int_X f \, \mathrm{d}\lambda$.

e. Show that $\lambda \ll \mu$, and hence find a g such that $\Lambda(f) = \int_X fg \, \mathrm{d}\mu$.

f. Show that the g chosen was in $L^q(X, \mathbb{R})$

g. Show that you can decompose a $\Lambda \in (L^p(X, \mathbb{R}))^*$ as a positive and negative part, using the idea of d, and hence that T is surjective.

There is a theorem called the open mapping theorem, that garuntees the inverse will also be bounded. Hence T is an isometric linear isomorphism.

h. Finally prove this result for $L^p(X, \mathbb{C})$ by extending from the real case.

Week 6

Last week, we read chapter 7 for this.

Problem 1

This problems will need the use of nets extensively, as we are leaving the world of metric spaces. I would suggest reading up on them somewhere, I recommend using Pederson's "analysis now" for it.

a. Let $\mathfrak{X}$ be an infinite dimensional Banach space. show the unit ball $\{x \in \mathfrak{X} : ||x|| \leq 1\}$ is not compact in the norm topology.

This is obviously not ideal. Fortunately, for dual spaces $\mathfrak{X}^*$, there is an alternate topology.

b. For a set X, and a set of functions $\mathcal{F}$ which are $f : X \longrightarrow Y$ for topological spaces Y_f, we can define the **Weak Topology** of $(X, \mathcal{F})$ as the weakest(= coarsest =least open sets) topology on X making each function of $\mathcal{F}$ continuous. Show that this topology always exists.

c. Let $\mathfrak{X}^*$ be the dual of Banach space $\mathfrak{X}$. We define the **Weak*-topology** on it as the weak topology generated by the functions

$$\mathrm{ev}_x : \mathfrak{X}^* \longrightarrow \mathbb{C}, \quad \mathrm{ev}_x(L) = L(x)$$

for $x \in \mathfrak{X}$. Show that $L_\lambda \to L$ in the weak*-topology iff $L_\lambda \to L$ pointwise as functions on $\mathfrak{X}$. Hence this can also be called the topology of pointwise convergence. (Be careful to use nets here, as outside of metric spaces sequences do not suffice.)

d. Finally prove **Banach-Alaoglu** theorem, that the unit ball of $\mathfrak{X}^*$ is compact in the weak*-topology. (Again use nets, the diagonalization of nets is not as easy as that of sequences. Partial credit for only considering sequences.)

Obviously the connection of this to Riesz-Markov is as follows, the space of finite complex Baire measures $M(X)$ is the dual of $C_0(X)$ for LCH X. We saw the total variation norm on this, which is also the operator norm. But it turns out to be more natural to work with the weak*-topology here

Problem 2

We will see an application of the weak* topology on the space of measures, this time in ergodic theory. So for a subset $A \subset \mathbb{N}$, we define the upper density as

$$d(A) = \limsup_{n\to\infty} \frac{\mathrm{card}(\{1, \ldots n\} \cap A)}{n}.$$

We want to prove

Theorem (Szemerédi). *If A has $d(A) > 0$, then it has arbritarily long arithmetic progressions.*

We will not actually prove this, but rephrase this in ergodic theoretic terms.

Let $W_2 = \{0,1\}^{\mathbb{N}}$. In our convention $\mathbb{N}$ has 0. This is a compact space because of Tychonoffs. Identify this with $P(\mathbb{N})$ in the obvious way. Let $\tau : W_2 \longrightarrow W_2$ denote the left shift

$$\tau(x_0, x_1, x_2 \dots) = (x_1, x_2 \dots).$$

We will work with the subspace $K := \overline{\{\tau^n A : n \in \mathbb{N}\}}$. Note that $\tau : K \longrightarrow K$.

a. Let $M := \{x \in K : x_0 = 1\}$. Show $\tau^n A \in M \iff n \in A$. Hence show that

$$a, a+n, \dots a+(k-1)n \in A \iff \tau^a A \in M \cap \tau^{-n}(M) \cdots \cap \tau^{-(k-1)n}(M).$$

From this deduce that Szemerédi's will be true if for each k, we can find a n such that $M \cap \tau^{-n}(M) \cdots \cap \tau^{-(k-1)n}(M)$ is non-empty.

The idea is now to make this into a measure theory problem, so we find some measure under which the set mentioned has positive measure, so it is non-empty.

b. Find a sequence of probability measures such that $\mu_i(M) = \dfrac{\text{card}(\{1, \dots i\} \cap A)}{i}$. Construct out of this (using the weak*-topology) a probability measure μ such that $\mu(M) = d(A)$. (read below)

It turns out W_2 also has a metric on it, which in turn means the unit ball of the space of measures is actually metrizable! So you can use sequences instead of nets.

c. Suppose $(X, \mathcal{M})$ is a measurable space and $T : X \longrightarrow X$ a measurable function. Let ν_n be a sequence of T invariant measure, i.e
$\nu_n(T^{-1}B) = \nu_n(B)$ for $B \in \mathcal{M}$. Let $\nu_n \to \nu$ weakly, then ν is also T invariant.

So with all this, we have a measure preserving system $(K, \mu; \tau)$ basically. What we did is called the **Furstenberg's correspondence**. It is a theorem due to him that

Theorem. *In an ergodic measure preserving system, if* $\mu(M) > 0$ *then*

$$\mu(M \cap \tau^{-n}(M) \cdots \cap \tau^{-(k-1)n}(M)) > 0.$$

We will not go into how to extract an ergodic measure from the μ we got. However, the theorem above is intuitive, ergodic systems are systems that "mix well", and so you expect that statement to be true.