There is a non-Connes embeddable Equivalence Relation

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What is Connes Embedding I

Definition

A group Γ is **Maximally Almost Periodic** if for each $g \neq 1 \in \Gamma$ there exists a homomorphism $\phi_g : \Gamma \to U_{n_m}$ with $\phi_g(g) \neq 1$.

- Equivalently, there is an embedding $G \hookrightarrow \prod_n U_n$.
- Lifts homomorphism invariant properties from finite groups.
- Let $\Gamma = \langle a, b, c, d \mid b^a = b^2,\ c^b = c^2,\ d^c = d^2,\ a^d = a^2 \rangle$. All finite dimensional reps of Γ are trivial.

What is Connes Embedding II

- Replace homomorphisms with **approximate homomorphisms**.
- An approximate homomorphism is a sequence $\phi_n : \Gamma \to U_{k_n}$ with

$$\left\|\phi_n(gg') - \phi_n(g)\phi_n(g')\right\|_2 \to 0 \quad \forall g, g' \in \Gamma.$$

- If approximate representations separate points of G, it is **Connes embeddable**.
- This is a property of the group von Neumann algebra $L(G)$ and generalizes to tracial von Neumann algebras.
- Replace U_n with $S_n \subset U_n$ to obtain **sofic** groups.

What is Connes Embedding III

- Equivalently $M \hookrightarrow \prod_U M_n(\mathbb{C})$ or $M \hookrightarrow R^U$ trace-preservingly.
- Can lift **first-order universal sentences** from finite-dimensional algebras.

Theorem (Ji–Natarajan–Vidick–Wright–Yuen '20)

There is a II_1 factor that is not Connes embeddable.

- Goal: Narrow down this II_1 factor.
- Are all $L(\Gamma)$ Connes embeddable?

Theorem (M '25)

There is an ergodic p.m.p. action $\Gamma \overset{\alpha}{\curvearrowright} (X, \mu)$ such that $L(\alpha)$ is not Connes embeddable.

Why is this important?

- The Connes Embedding Problem (CEP) was equivalent to several central problems:
  - Tsirelson's problem
  - The Microstates conjecture
  - The Kirchberg conjecture
- Sofic groups are known to satisfy important properties:
  - Lück's Determinant Conjecture
  - The Algebraic Eigenvalue Conjecture
  - Kervaire–Laudenbach Conjecture
  - Direct Finiteness
  - Gottschalk's Surjunctivity Conjecture
  - can define entropy

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Traces on group

- A **trace on** Γ is a function $\tau : \Gamma \to \mathbb{C}$ such that:
  - τ is **positive definite**: $\sum_{i,j} c_i \overline{c_j}\, \tau(g_i^{-1} g_j) \geq 0$.
  - τ is **conjugation invariant**: $\tau(hgh^{-1}) = \tau(g)$.
  - $\tau(e) = 1$.
- To each $\tau \in T(\Gamma)$ one can associate a tracial von Neumann algebra $\tau(\Gamma)''$, called the GNS.

Reduction to MIP*=RE I

- Let M be a separable tracial von Neumann algebra.
- There is a trace τ on $\mathbb{F}_\infty$ with $\tau(\mathbb{F}_\infty)'' = M$.
- M is Connes embeddable precisely when τ is an **amenable trace**.
- For $\mathbb{F}_\infty$ this means $\tau = \lim \tau_n$ where τ_n have finite-dimensional GNS.
- CEP reduces to whether $T(\mathbb{F}_\infty) = AT(\mathbb{F}_\infty)$.

Reduction to MIP*=RE II

- Let $A \subset B$ be convex sets; let I be a countable index set.
- $T(\mathfrak{G}, -) : B \to [0,1]$ for $\mathfrak{G} \in I$ are affine maps.
- $\omega^{co}(\mathfrak{G}) = \sup_{b \in B} T(\mathfrak{G}, b), \quad \omega^*(\mathfrak{G}) = \sup_{a \in A} T(\mathfrak{G}, a)$.
- There is a computably enumerable sequence $\omega_d^*(\mathfrak{G}) \nearrow \omega^*(\mathfrak{G})$.
- There is a computably enumerable sequence $\omega_d^{co}(\mathfrak{G}) \searrow \omega^{co}(\mathfrak{G})$.
- If $\omega^*(\mathfrak{G})$ is uncomputable, then $A \neq B$.

Reduction to MIP*=RE III

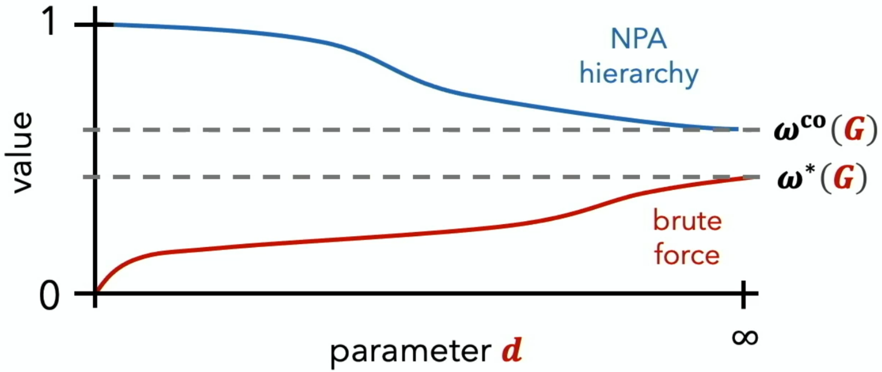

Figure: credit: John Wright

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Invariant random subgroups

- Let Γ be a discrete countable group.
- Let $\mathrm{sub}(\Gamma) \subset \{0,1\}^\Gamma$ be the set of subgroups of Γ.
- A **Random Subgroup** is a Borel measure $H \in \mathrm{Prob}(\mathrm{sub}(\Gamma))$.
- An **Invariant Random Subgroup (IRS)** is a random subgroup invariant under the conjugation action of Γ on $\mathrm{sub}(\Gamma)$.

Examples of IRS

1. If $N \lhd \Gamma$ is normal, then δ_N is an IRS.
2. If α is a p.m.p. action of Γ on (X, μ), define $\nu_\alpha \in \mathrm{IRS}(\Gamma)$ by

$$\nu_\alpha(A) = \mu\big(\{x \in X : \ \mathrm{Stab}(x) \in A\}\big), \quad A \subset \mathrm{sub}(\Gamma).$$

- Pick $x \in X$ randomly and look at $\mathrm{Stab}(x)$.
- If we consider $\mathrm{Stab} : X \to \mathrm{sub}(\Gamma)$ then $\nu_\alpha = \mathrm{Stab}_*(\mu)$.

Examples of IRS II

Theorem (Albért-Glasner-Virág'14)

All IRS of countable groups arise as above.

3. If $H \in \mathrm{IRS}(\Gamma)$ arises from an action on a finite set, then H is **finitely described**.
4. If $H \in \mathrm{IRS}(\Gamma)$ is the weak* limit of finitely described IRS, then H is **co-sofic**.

proposition

Let $N \lhd \Gamma$ where Γ is a free group. Then δ_N is co-sofic $\iff$ Γ/N is sofic.

Aldous–Lyons

Theorem (Bowen–Chapman–Lubotzky–Vidick '24)

There is a non co-sofic IRS on any free group.

- $\mathrm{IRS}(\Gamma) \neq \mathrm{IRS}_{sof}(\Gamma)$.
- Their proof reduces to subgroup tests and then to $\mathrm{MIP}^*{=}\mathrm{RE}$, which is very cumbersome.
- Using operator-algebraic methods, we can get this directly without subgroup test.
- We still use their uncomputability result as a black box.

Quotienting by an IRS

- For $H \in \mathrm{IRS}(\Gamma)$ define a trace $\tau_H(g) := \mathbb{P}(g \in H)$.
- If H arises from an action of Γ on (X, μ) then $\tau_H(g) = \mu(gx = x)$.

proposition

If $H = \delta_N$, then $\tau_H(\Gamma)'' = L(\Gamma/N)$.

Definition

For $H \in \mathrm{IRS}(\Gamma)$, define $L(\Gamma/H) := \tau_H(\Gamma)''$.

proposition

If H comes from $\alpha : \Gamma \curvearrowright (X, \mu)$ then $L(\Gamma/H) \subset L(\mathcal{R}_\alpha)$.

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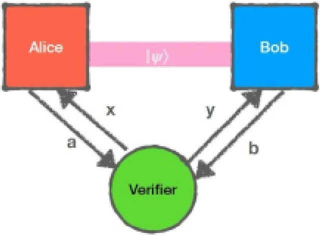
Non-Local Games



- Finite question set Q and answer set A.
- Verifier samples question pair (x, y) from $Q \times Q$ according to π.
- Alice and Bob answer $(a, b) \in A \times A$.
- Verifier accepts if $V(a, b \mid x, y) = 1$.

Strategies

- A strategy (correlation) is the conditional distribution $(p(a, b \mid x, y))_{a,b,x,y}$.
- A **Projection-Valued Measure (PVM)** is a collection of projections e_x^a $(x \in Q,\ a \in A)$ with

$$\sum_a e_x^a = 1, \qquad e_x^a e_x^b = \delta_{ab} e_x^a.$$

- Think of e_x^a as quantum probabilities.
- The universal C^*-algebra generated by $\{e_x^a\}$ is $C^*(\mathbb{F}_{Q,A})$.

Strategies II

- Strategies come from states on $C^*(\mathbb{F}_{Q,A})$ with

$$p(a, b \mid x, y) = \tau(e_x^a e_y^b).$$

- If the states are traces, these are **quantum commuting** strategies.
- If the states are finite-dimensional traces, these are **quantum** strategies.
- If the states are amenable traces then these are **quantum approximate**.

Stategies III

- Let $\omega^*(\mathfrak{G})$ be the best winning probability with a quantum approximate strategy.
- ω^{co} is same for quantum commuting strategies.
- MIP*=RE proves ω^* is uncomputable.

IRS strategies

Definition

An **IRS strategy** on $\mathfrak{G}$ is a strategy coming from a trace τ on $C^*(\mathbb{F}_{Q,A}) \otimes C^*(\mathbb{Z}/2\mathbb{Z})$ with

$$p(a, b \mid x, y) = \tau\big((1 - J)e_x^a e_y^b\big).$$

A **permutation strategy** is an IRS strategy coming from a finite-dimensional IRS.

Theorem (M '25)

There is a game $\mathfrak{G}$ with $\omega_{\mathrm{IRS}}(\mathfrak{G}) > \omega^*(\mathfrak{G})$.

So there is a non-amenable trace τ on a free group coming from an IRS.

The computability result

Theorem (Bowen, Chapman, Vidick '24)

There is a computable map taking Turing machines to non-local games, $M \mapsto \mathfrak{G}_M$, such that:

- *If M halts, then $\mathfrak{G}_M$ has a perfect permutation strategy.*
- *If M does not halt, then $\omega^*(\mathfrak{G}_M) < 1/2$.*

Advantages of Undecidability

- Criticism: This procedure does not produce explicit examples.
- In follow up work, I proved there is an IRS satisfying Lück's determinant conjecture that is not co-sofic using undecidability.
- Lyons-Terlov-Vidyánszky managed to prove ineffectiveness of the regularity lemma for bounded degree graphs using undecidability.