Model Theory of II_1 factors

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- Classical Model theory is a widely useful tool in a lot of fields of math, including algebraic geometry and combinatorics.
- Continuous Model theory turns out to be extremely useful

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  - $(\forall x(x \cdot x^{-1} = e)) \wedge \neg(y = e))$ is a L_{Group}-formula.

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- All of this can be multi-sorted.
  - The language of Vector spaces should involve scalar multiplication, which is some function $F \times V \to V$. This is a multi-sorted function.

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- $\forall, \exists$ replaced by $\sup, \inf$.

Language for Tracial von Neumann Algebras

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- finally for $m > n$, an inclusion $i_{m,n} : B_n \to B_m$.

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- For II_1 factors, consider projections of the same trace.

Ultraproducts

Definition

An ultraproduct of L-structures $\{\mathfrak{M}_i\}_I$ over ultrafilter $\mathcal{U}$ on I is the metric structure

$$\int_I \mathfrak{M}_i \mathrm{d}\mathcal{U} = \overline{\prod_I \mathfrak{M}_i / \sim}^{\lim_{\mathcal{U}} d}$$

where $x \sim y$ if $\lim_U d_i(x, y) = 0$. The functions/constants/relations are interpretated pointwise, and the metric is the ultralimit.

Łós' Theorem

Theorem (Łós)

Let $\mathfrak{M} = \int \mathfrak{M}_i \mathrm{d}\mathcal{U}$ *be an ultraproduct of L-structures, then for any sentence* σ*:*

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If every finite $T' \subset T$ *has an approximate model, then* T *has a model.*

Proof.

Let $\mathfrak{M}_F$ be a model corresponding to finite subtheory F so that $|\sigma_F^{\mathfrak{M}}| \leq 1/|F|$. An appropriate ultraproduct $\int \mathfrak{M}_F \mathrm{d}\mathcal{U}$ will be a model of all of T. □

Elementary equivalence

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We say that two L-structures $\mathfrak{M}$ and $\mathfrak{M}$ are elementarily equivalent (denoted $\mathfrak{M} \equiv \mathfrak{N}$) if for each sentence σ, $\sigma^{\mathfrak{M}} = \sigma^{\mathfrak{N}}$.

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Theorem (Keisler-Shelah, Farah-Hart-Sherman)

$M \equiv N \iff$ *there is an ultrafilter* $\mathcal{U}$ *so that* $\mathfrak{M}^{\mathcal{U}} \cong \mathfrak{N}^{\mathcal{U}}$.

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Theorem

A class $\mathcal{C}$ of L-structure is axiomitizable $\iff$ $\mathcal{C}$ is closed under ultraproducts and elementary equivalence. Elementary equivalence can be replaced by Ultraroots because of Kiesler-Shelah.

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Theorem (Murray-von Nuemann)

$L(F_2) \not\cong \mathcal{R}$

Proof.

$\mathcal{R}$ has property (Γ) while $L(F_2)$ doesn't. □

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M does not have property (Γ) $\iff$ there is a constant C and a finite set $F \subset U(M)$, there is a y so that $||y - \mathrm{tr}(y)||_2 \leq C \sum_{x \in F} ||[x, y]||_2$. □

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- it is known by deep results of Kharlampovich-Miyasnikov and Sela that $F_n \equiv F_m$ is the language of groups.
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- $G \equiv H$ does not imply $L(G) \equiv L(H)$.
    - Goldbring showed counterexamples of $\text{Sl}_2(\mathbb{F}_2^{alg})$ and $\text{Sl}_2(\mathbb{F}_2(t)^{alg})$.

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- Question: are there any factors such that $\mathcal{F}^{fo}(M) \neq \mathbb{R}^+$?

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Free Factor Isomorphism problem

- There are interpolated Free group factors $L(F_r)$, $r \in [1, \infty]$

Theorem (Râdelescu)

One of the following holds:

1. $L(F_r) \cong L(F_s)$ *for all* $1 < r \leq s \leq \infty$ *and* $\mathcal{F}(L(F_r)) = \mathbb{R}^+$.
2. $L(F_r) \ncong L(F_s)$ *for all* $1 < r \neq s \leq \infty$ *and* $\mathcal{F}(L(F_r)) = \{1\}$.

Free Factor Elementary Equivalence Problem

Theorem (Goldbring-Pi)

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- In addition Goldbring-Pi also proves that if $\mathcal{F}^{fo}(F_r)$ arent trivial, then every $\forall\exists$ sentence takes the same value in all the $L(F_r)$.

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- In addition Goldbring-Pi also proves that if $\mathcal{F}^{fo}(F_r)$ arent trivial, then every $\forall\exists$ sentence takes the same value in all the $L(F_r)$.
- By Sela, every sentence in F_n can be reduced to a conjunction of $\forall\exists$ sentences.

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