THE FIRST AND SECOND VARIATIONS OF THE LENGTH-INTEGRAL IN RIEMANNIAN SPACE

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Synopsis: 1. Introduction. 2. Notation. 3. First form of the first variation. 4. Second form of the first variation. 5. First form of the second variation. 6. Second form of the second variation. 7. Introduction of the Riemannian curvature. 8. Second variation when the curve is geodesic and the end-points are fixed. 9. Expression for the second variation using the unit normal variation vector. 10. Conjugate points.

1. *Introduction.*

The problem of the variations of the length-integral in Riemannian space is a special case of the general Lagrangian problem*. The present paper is, however, developed independently of the existing general theory, the methods being based essentially on the theory of tensors. Extensive use is made of the contravariant form associated with the parallel propagation of Levi-Civita†.

We consider a manifold V_N in which there exist a coordinate system $(x^1, x^2, \ldots, x^N)$ and a line-element ds defined by

$$ds^2 = g_{mn}\, dx^m\, dx^n, \tag{1.1}$$

where the right-hand side is a positive definite form and $g_{mn}\,(= g_{nm})$ are functions of the coordinates only, possessing continuous partial deriva-

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* O. Bolza, *Vorlesungen über Variationsrechnung* (1909), 6, and ch. 11 and 12.

† See L. Bianchi, *Lezioni di Geometria Differenziale*, 2_2 (1924), 790.

tives of the second order with respect to the coordinates at all points of a region S of V_N.

Let C be a curve joining the points P_1 and P_2 and lying wholly in S. Let C be defined by the equations

$$(1.2) \qquad x^i = x^i(u),$$

where $u = u_1$ at P_1 and $u = u_2$ at P_2, u_2 being greater than u_1. The length of this curve is by definition

$$(1.3) \qquad L = \int_{u_1}^{u_2} \left\{ g_{mn} \frac{dx^m}{du} \frac{dx^n}{du} \right\}^{\frac{1}{2}} du.$$

We shall consider only those curves for which d^2x^i/du^2 are continuous $(u_1 \leqslant u \leqslant u_2)$ and for which dx^i/du do not all vanish simultaneously for any value of u in this range.

The equations

$$(1.4) \qquad x^i = x^i(u, v)$$

define a two-space V_2. Let

$$(1.5) \qquad x^i(u, 0) \equiv x^i(u) \quad (u_1 \leqslant u \leqslant u_2),$$

so that the curve $v = 0$ of V_2 coincides with C. We shall call the portions of the parametric lines of u intercepted between the curves $u = u_1$ and $u = u_2$ the *u-curves* or *varied positions of* C, and the parametric lines of v the *v-curves* or *curves of variation*. The curves of variation of P_1 and P_2 have the equations $u = u_1$ and $u = u_2$ respectively. We shall assume the functions in (1.4) to be such that

$$\frac{\partial^2 x^i}{\partial u^2}, \quad \frac{\partial^2 x^i}{\partial u\, \partial v}, \quad \frac{\partial^2 x^i}{\partial v^2}, \quad \frac{\partial^3 x^i}{\partial u^2 \partial v}, \quad \frac{\partial^3 x^i}{\partial u\, \partial v^2}$$

are continuous functions of u and v for $u_1 \leqslant u \leqslant u_2$ and for a small range of values of v on either side of zero. We shall further assume that $(\partial x^i/\partial v)_{v=0}$ do not all vanish simultaneously for more than a finite number of values of u between u_1 and u_2. The length of a u-curve is a function of v only, since we are not considering variations of u_1 and u_2; we shall write this length $L(v)$. The parameter u on C will be chosen equal to the length of the arc of C measured from some definite point, so that $du/ds = 1$ on C. This choice of parameter is important, since it produces considerable simplification in the formulae.

2. *Notation.*

We shall indicate the partial derivative with respect to u of any function of u and v by a superposed point,

$$\frac{\partial X}{\partial u} = \dot{X}, \tag{2.01}$$

and the partial derivative with respect to v by a dash,

$$\frac{\partial X}{\partial v} = X'. \tag{2.02}$$

We shall write $\xi^i = \dot{x}^i$, $\eta^i = x^{i\prime}$, so that

$$\xi^{i\prime} = \dot{\eta}^i. \tag{2.10}$$

We shall call η^i the *variation vector* and $\eta^i \delta v$ the *infinitesimal variation vector*. The *magnitude* of a vector X^i will be denoted by X, so that

$$X^2 = g_{mn} X^m X^n \quad (X \geqslant 0). \tag{2.11}$$

A *unit vector* being one whose magnitude is unity, we shall call μ^i the *unit variation vector* where

$$\eta^i = \eta \mu^i, \tag{2.12}$$

μ^i being therefore codirectional with η^i. The angle between two vectors X^i and Y^i will be denoted by $\theta(\mathbf{X}, \mathbf{Y})$, so that

$$XY \cos \theta(\mathbf{X}, \mathbf{Y}) = g_{mn} X^m Y^n. \tag{2.125}$$

The length of any u-curve is

$$L(v) = \int_{u_1}^{u_2} \{g_{mn} \xi^m \xi^n\}^{\frac{1}{2}} du, \tag{2.13}$$

or, writing

$$F^2 = g_{mn} \xi^m \xi^n, \tag{2.14}$$

$$L(v) = \int_{u_1}^{u_2} F\, du. \tag{2.15}$$

We note that $F = 1$ along any u-curve for which $du/ds = 1$. We shall denote the partial derivatives of F with respect to x^i and ξ^i in the following manner :—

$$(2.16)\qquad \begin{cases} \dfrac{\partial F}{\partial x^i} = F_{i|}, \quad \dfrac{\partial F}{\partial \xi^i} = F_{|i}, \\ \dfrac{\partial^2 F}{\partial x^i \partial x^j} = F_{ij|}, \quad \dfrac{\partial^2 F}{\partial x^i \partial \xi^j} = F_{i|j}, \quad \dfrac{\partial^2 F}{\partial \xi^i \partial \xi^j} = F_{|ij}. \end{cases}$$

It is to be noted that only those expressions which have all their subscripts to the right of the vertical bar are tensors*. We shall now write the explicit expressions for the partial derivatives in the case where $F = 1$ on the u-curve in question. We find

$$(2.17)\qquad F_{i|} = \frac{1}{2F}\frac{\partial(F^2)}{\partial x^i} = \tfrac{1}{2}\frac{\partial g_{jk}}{\partial x^i}\xi^j\xi^k,$$

$$(2.18)\qquad F_{|i} = \frac{1}{2F}\frac{\partial(F^2)}{\partial \xi^i} = g_{ij}\xi^j,$$

$$(2.19)\qquad F_{ij|} = \frac{\partial}{\partial x^i}\left(\frac{1}{2F}\frac{\partial(F^2)}{\partial x^j}\right) = \frac{1}{2F}\frac{\partial^2(F^2)}{\partial x^i \partial x^j} - \frac{1}{4F^3}\frac{\partial(F^2)}{\partial x^i}\frac{\partial(F^2)}{\partial x^j}$$

$$= \tfrac{1}{2}\frac{\partial^2 g_{kl}}{\partial x^i \partial x^j}\xi^k\xi^l - \tfrac{1}{4}\frac{\partial g_{kl}}{\partial x^i}\xi^k\xi^l\frac{\partial g_{mn}}{\partial x^j}\xi^m\xi^n,$$

$$(2.20)\qquad F_{i|j} = \frac{\partial}{\partial x^i}\left(\frac{1}{2F}\frac{\partial(F^2)}{\partial \xi^j}\right) = \frac{1}{2F}\frac{\partial^2(F^2)}{\partial x^i \partial \xi^j} - \frac{1}{4F^3}\frac{\partial(F^2)}{\partial x^i}\frac{\partial(F^2)}{\partial \xi^j}$$

$$= \frac{\partial g_{jk}}{\partial x^i}\xi^k - \tfrac{1}{2}\frac{\partial g_{kl}}{\partial x^i}\xi^k\xi^l g_{jm}\xi^m,$$

$$(2.21)\qquad F_{|ij} = \frac{\partial}{\partial \xi^i}\left(\frac{1}{2F}\frac{\partial(F^2)}{\partial \xi^j}\right) = \frac{1}{2F}\frac{\partial^2(F^2)}{\partial \xi^i \partial \xi^j} - \frac{1}{4F^3}\frac{\partial(F^2)}{\partial \xi^i}\frac{\partial(F^2)}{\partial \xi^j}$$

$$= g_{ij} - g_{ik}\xi^k g_{jl}\xi^l.$$

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* Cf. "A generalization of the Riemannian line element", *Trans. Amer. Math. Soc.* 27 (1925), 61. What was denoted by F in that paper is here denoted by F^2.

If X^i be a contravariant vector given along a curve $x^i = x^i(t)$, then, as is well known*,

$$\frac{dX^i}{dt} + \left\{ \begin{matrix} m & n \\ & i \end{matrix} \right\} X^m \frac{dx^n}{dt} \tag{2.22}$$

is also a contravariant vector: it may conveniently be termed the *contravariant derivative of X^i with respect to the given curve.* We shall denote the contravariant derivative of a vector with respect to a u-curve by a superposed bar,

$$\bar{X}^i = \dot{X}^i + \left\{ \begin{matrix} m & n \\ & i \end{matrix} \right\} X^m \xi^n, \tag{2.23}$$

and with respect to a v-curve by a superposed circumflex accent,

$$\hat{X}^i = X^{i\prime} + \left\{ \begin{matrix} m & n \\ & i \end{matrix} \right\} X^m \eta^n. \tag{2.24}$$

When two of these operational symbols occur together, the order is to be read from the top downwards. By (2 . 10) it is easily seen that

$$\hat{\xi}^i = \bar{\eta}^i. \tag{2.25}$$

The equations for parallel propagation of X^i along the u-curves and the v-curves are respectively $\bar{X}^i = 0$ and $\hat{X}^i = 0$.

3. *First form of the first variation.*

The first variation of the length of a u-curve for the displacement arising from an infinitesimal increment δv is by definition $\delta L = L'(v)\delta v$. We have from (2 . 15)

$$L'(v) = \int_{u_1}^{u_2} F' \, du. \tag{3.1}$$

Now

$$\begin{aligned} F' &= F_{|i}\, \xi^{i\prime} + F_{i|}\, x^{i\prime} \\ &= F_{|i}\, \dot{\eta}^i + F_{i|}\, \eta^i \\ &= F_{|i}\left(\bar{\eta}^i - \left\{ \begin{matrix} j & k \\ & i \end{matrix} \right\} \eta^j \xi^k\right) + F_{i|}\, \eta^i \\ &= F_{|i}\, \bar{\eta}^i + \Psi_i\, \eta^i, \end{aligned} \tag{3.11}$$

where

$$\Psi_i = F_{i|} - F_{|l} \left\{ \begin{matrix} i & k \\ & l \end{matrix} \right\} \xi^k. \tag{3.12}$$

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* Cf. Bianchi, *loc. cit.*, 790.

Inspection of (3 . 11) shows that Ψ_i is a covariant vector. But if we introduce a system of Riemannian coordinates* at the point in question, so that $\partial g_{jk}/\partial x^l$ all vanish at the point, we see that all the components of Ψ_i vanish. Therefore, from the covariant character, they must also vanish for any coordinate system. Thus we have

$$F' = F_{|i}\bar{\eta}^i. \tag{3.2}$$

Hence, for any value of v,

$$L'(v) = \int_{u_1}^{u_2} F_{|i}\bar{\eta}^i\,du. \tag{3.3}$$

To evaluate this expression for the curve C, we may apply (2 . 18) and obtain the equivalent forms for the first variation,

$$\delta L = \delta v \int_{u_1}^{u_2} g_{ij}\,\xi^i\,\bar{\eta}^j\,du, \tag{3.4}$$

$$\delta L = \delta v \int_{u_1}^{u_2} \bar{\eta}\cos\theta(\boldsymbol{\xi}, \bar{\boldsymbol{\eta}})\,du. \tag{3.5}$$

We shall call either of these expressions the *first form of the first variation.*

The following theorems result directly from (3 . 5) :

THEOREM I.—*The first variation of the length of a curve C is zero when the variation vector is propagated parallelly along C.*

In this case C and the varied curve cannot have a common point, for in parallel propagation the magnitude of a vector remains constant. Thus if the magnitude of the variation vector vanishes at one point of C, it vanishes at all points, and there is no infinitesimal displacement.

THEOREM II.—*The first variation of the length of a curve C is zero if the contravariant derivative of the variation vector with respect to C is normal to C at every point.*

THEOREM III.—*The first variation of the length of a curve C has the same sign as (the opposite sign to) δv, if the contravariant derivative of the variation vector with respect to C makes an acute (obtuse) angle with C at every point.*

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* Cf. Laue, *Relativitätstheorie*, 2 (1921), 75.

4. *Second form of the first variation.*

Since

$$\frac{\partial}{\partial u}(g_{ij}\,\xi^i\,\eta^j) = g_{ij}\,\bar{\xi}^i\,\eta^j + g_{ij}\,\xi^i\,\bar{\eta}^j, \tag{4.1}$$

we find at once from (3 . 4) for the curve C

$$\delta L = \delta v\left([g_{ij}\,\xi^i\,\eta^j]_{u_1}^{u_2} - \int_{u_1}^{u_2} g_{ij}\,\bar{\xi}^i\,\eta^j\,du\right), \tag{4.2}$$

or

$$\delta L = \delta v\left([\eta\cos\theta(\boldsymbol{\xi},\boldsymbol{\eta})]_{u_1}^{u_2} - \int_{u_1}^{u_2} \bar{\xi}\eta\cos\theta(\bar{\boldsymbol{\xi}},\boldsymbol{\eta})\,du\right). \tag{4.3}$$

We shall call either of these expressions the *second form of the first variation.*

Noting that the vector $\bar{\xi}^i$ defines the principal normal of C and that $\bar{\xi}$ is the principal curvature*, and also that $\bar{\xi}$ vanishes if C is geodesic, the following theorems result directly, Theorem IV being well known :

THEOREM IV.—*The first variation of the length of a curve C is zero if C is geodesic and the variation vector either vanishes or is normal to C at the end points.*

THEOREM V.—*The first variation of the length of a curve C is zero if the variation vector is perpendicular to the principal normal of C at every point and either vanishes or is normal to C at the end points.*

THEOREM VI.—*The first variation of the length of a curve C has the same sign as (the opposite sign to) δv if the variation vector is perpendicular to the principal normal of C at every point and makes with the direction of C at the end points angles which are obtuse and acute (acute and obtuse) in order.*

The principal normal of a curve has an intrinsic positive sense defined by $\bar{\xi}^i$, since a reversal of the sense of the curve in which u increases does

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* Cf. Bianchi, *loc. cit.*, 456.

not reverse $\bar{\xi}^i$. We may state as a direct deduction from (4 . 2) the following theorem :

THEOREM VII.—*The first variation of the length of a curve C is negative for an infinitesimal variation in the positive direction of the principal normal of C at every point.*

5. *First form of the second variation.*

The second variation of the length of a u-curve for the displacement arising from an infinitesimal increment δv is by definition

$$\delta^2 L = \tfrac{1}{2} L''(v)\delta v^2.$$

To find $L''(v)$ we differentiate (3 . 1) with respect to v, obtaining

$$L''(v) = \int_{u_1}^{u_2} F'' du, \tag{5.10}$$

where, by (3 . 2),

$$\begin{aligned} F'' &= \frac{\partial}{\partial v}(F_{|i}\bar{\eta}^i) \\ &= (F_{|ij}\xi^{j\prime} + F_{j|i}\eta^j)\bar{\eta}^i + F_{|i}\left(\hat{\bar{\eta}}^i - \left\{ {j\ k \atop i} \right\} \bar{\eta}^j \eta^k\right). \end{aligned} \tag{5.11}$$

Hence, since

$$\xi^{j\prime} = \dot{\eta}^j = \bar{\eta}^j - \left\{ {k\ l \atop j} \right\} \eta^k \xi^l, \tag{5.12}$$

we find

$$F'' = F_{|i}\hat{\bar{\eta}}^i + F_{|ij}\bar{\eta}^i\bar{\eta}^j + \Phi_{ij}\bar{\eta}^i\eta^j, \tag{5.13}$$

where

$$\Phi_{ij} = -F_{|ik}\left\{ {j\ l \atop k} \right\}\xi^l + F_{j|i} - F_{|k}\left\{ {i\ j \atop k} \right\}. \tag{5.14}$$

Inspection of (5 . 13) shows that Φ_{ij} is a covariant tensor. But if we introduce a system of Riemannian coordinates at the point in question, it is easy to see that all the components of Φ_{ij} vanish. Therefore from the covariant character they must also vanish for every coordinate system, and we have

$$F'' = F_{|i}\hat{\bar{\eta}}^i + F_{|ij}\bar{\eta}^i\bar{\eta}^j. \tag{5.15}$$

Substitution in (5 . 10) gives

$$L''(v) = \int_{u_1}^{u_2} (F_{|i}\hat{\eta}^i + F_{|ij}\bar{\eta}^i\bar{\eta}^j)\,du. \tag{5.16}$$

For the curve C we find, using (2 . 18) and (2 . 21),

$$F_{|i}\hat{\eta}^i + F_{|ij}\bar{\eta}^i\bar{\eta}^j = g_{ij}\xi^i\hat{\eta}^j + g_{ij}\bar{\eta}^i\bar{\eta}^j - g_{ik}\xi^k g_{jl}\xi^l\bar{\eta}^i\bar{\eta}^j$$

$$= g_{ij}\xi^i\hat{\eta}^j + g_{ij}\bar{\eta}^i\bar{\eta}^j - (g_{ij}\xi^i\bar{\eta}^j)^2 \tag{5.17}$$

$$= \hat{\eta}\cos\theta(\boldsymbol{\xi}, \hat{\boldsymbol{\eta}}) + \bar{\eta}^2\sin^2\theta(\boldsymbol{\xi}, \bar{\boldsymbol{\eta}}). \tag{5.18}$$

Substitution in (5 . 16) gives the equivalent forms

$$\delta^2 L = \tfrac{1}{2}\delta v^2 \int_{u_1}^{u_2} [g_{ij}\xi^i\hat{\eta}^j + g_{ij}\bar{\eta}^i\bar{\eta}^j - (g_{ij}\xi^i\bar{\eta}^j)^2]\,du, \tag{5.19}$$

$$\delta^2 L = \tfrac{1}{2}\delta v^2 \int_{u_1}^{u_2} [\hat{\eta}\cos\theta(\boldsymbol{\xi}, \hat{\boldsymbol{\eta}}) + \bar{\eta}^2\sin^2\theta(\boldsymbol{\xi}, \bar{\boldsymbol{\eta}})]\,du. \tag{5.20}$$

Either of these forms we shall call the *first form of the second variation*. They exist whether the first variation vanishes or not.

There are evidently some critical conditions governing the sign of the second variation. These we proceed to write down.

CONDITION (α).—*$\hat{\eta} = 0$ at every point of C.*

If the vector $\bar{\eta}^i$ is propagated parallelly along the v-curves, this condition is satisfied.

CONDITION $\left\{\begin{matrix}(\beta_1)\\(\beta_2)\\(\beta_3)\end{matrix}\right\}$.—*$\theta(\boldsymbol{\xi}, \hat{\boldsymbol{\eta}})$ is $\left\{\begin{matrix}\text{acute}\\\text{a right angle}\\\text{obtuse}\end{matrix}\right\}$ at every point of C.*

CONDITION (γ).—*$\bar{\eta} = 0$ at every point of C.*

This is a necessary and sufficient condition that the variation vector should be propagated parallelly along C.

CONDITION (δ).—*The vector* $\bar{\eta}^i$ *is codirectional with C wherever it does not vanish.*

The following theorems result directly from (5 . 20) :

THEOREM VIII.—*The second variation is positive if* (α) *is not true and* (β_1) *is true.*

THEOREM IX.—*The second variation is positive if one of the three conditions* (α), (β_1), (β_2) *is true and neither* (γ) *nor* (δ) *is true.*

THEOREM X.—*The second variation is zero if either* (α) *or* (β_2) *is true and either* (γ) *or* (δ) *is true.*

THEOREM XI.—*The second variation is negative if* (β_3) *and either* (γ) *or* (δ) *are true and* (α) *is not true.*

6. *Second form of the second variation.*

We shall now obtain a second form of the second variation by application of the method of integration by parts and by use of the following fact. If X^i be any contravariant vector given as a function of u and v, it is easily seen by direct computation that

$$\hat{\bar{X}}^i-\bar{\hat{X}}^i = -G^i_{jkl}X^j\xi^k\eta^l, \tag{6.10}$$

where

$$G^i_{jkl} = \frac{\partial}{\partial x^k}\begin{Bmatrix} j & l \\ & i \end{Bmatrix}-\frac{\partial}{\partial x^l}\begin{Bmatrix} j & k \\ & i \end{Bmatrix}+\begin{Bmatrix} j & l \\ & m \end{Bmatrix}\begin{Bmatrix} m & k \\ & i \end{Bmatrix}-\begin{Bmatrix} j & k \\ & m \end{Bmatrix}\begin{Bmatrix} m & l \\ & i \end{Bmatrix}, \tag{6.11}$$

the curvature tensor of V_N*. Thus

$$\hat{\bar{\eta}}^i-\bar{\hat{\eta}}^i = -G^i_{jkl}\eta^j\xi^k\eta^l, \tag{6.12}$$

and therefore

$$\int_{u_1}^{u_2}F_{|i}\hat{\bar{\eta}}^i\,du = \int_{u_1}^{u_2}F_{|i}\bar{\hat{\eta}}^i\,du-\int_{u_1}^{u_2}F_{|i}\,G^i_{jkl}\,\eta^j\xi^k\eta^l\,du. \tag{6.13}$$

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* Cf. J. A. Schouten, *Der Ricci-Kalkül* (1924), 83, where the differential notation is employed. Schouten's arrangement of indices is, however, different. The present notation follows Einstein, *Die Grundlage der allgemeinen Relativitätstheorie* (1916), 39, and Weyl, *Raum, Zeit, Materie* (1921), 107.

Substituting in (5 . 16) and using (2 . 18), we obtain for the curve C

$$(6.14)\qquad L''(0) = \int_{u_1}^{u_2} g_{ij}\,\xi^i \bar{\hat{\eta}}^j\,du + \int_{u_1}^{u_2} [g_{ij}\,\bar{\eta}^i\,\bar{\eta}^j - (g_{ij}\,\xi^i\,\bar{\eta}^j)^2 - G_{ijkl}\,\xi^i\,\eta^j\,\xi^k\,\eta^l]\,du,$$

where

$$(6.15)\qquad G_{ijkl} = g_{in}\,G^n_{jkl}.$$

Now

$$(6.16)\qquad \frac{\partial}{\partial u}(g_{ij}\,\xi^i\,\hat{\eta}^j) = g_{ij}\,\bar{\xi}^i\,\hat{\eta}^j + g_{ij}\,\xi^i\,\bar{\hat{\eta}}^j,$$

and therefore

$$(6.17)\qquad \int_{u_1}^{u_2} g_{ij}\,\xi^i\,\bar{\hat{\eta}}^j\,du = [g_{ij}\,\xi^i\,\hat{\eta}^j]_{u_1}^{u_2} - \int_{u_1}^{u_2} g_{ij}\,\bar{\xi}^i\,\hat{\eta}^j\,du.$$

Substituting in (6 . 14), we see that the second variation may be written

$$(6.18)\qquad \delta^2 L = \tfrac{1}{2}\delta v^2(T - I_1 + I_2),$$

where

$$(6.19)\qquad T = [\hat{\eta}\cos\theta(\boldsymbol{\xi}, \hat{\boldsymbol{\eta}})]_{u_1}^{u_2},$$

$$(6.20)\qquad I_1 = \int_{u_1}^{u_2} \bar{\xi}\hat{\eta}\,\cos\theta(\bar{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}})\,du,$$

$$(6.21)\qquad I_2 = \int_{u_1}^{u_2} [\bar{\eta}^2 \sin^2\theta(\boldsymbol{\xi}, \boldsymbol{\eta}) - G_{ijkl}\,\xi^i\,\eta^j\,\xi^k\,\eta^l]\,du.$$

We shall call (6 . 18) the *second form of the second variation*. It exists whether the first variation vanishes or not.

7. *Introduction of the Riemannian Curvature.*

We shall now modify the expression for I_2 by the introduction of the Riemannian curvature.

If, at any point of V_N, W_M $(M \leqslant N)$ is formed from all the geodesics of V_N emanating from the point in directions lying in an assigned

M-element, and if the equations of W_M are expressed parametrically as

(7 . 1) $$x^i = x^i(\underset{0}{x}^1, \underset{0}{x}^2, \ldots, \underset{0}{x}^M),$$

then

(7 . 2) $$\underset{0}{G}_{\mu\nu\sigma\tau} = G_{mnst} \frac{\partial x^m}{\partial \underset{0}{x}^\mu} \frac{\partial x^n}{\partial \underset{0}{x}^\nu} \frac{\partial x^s}{\partial \underset{0}{x}^\sigma} \frac{\partial x^t}{\partial \underset{0}{x}^\tau} \quad (\mu, \nu, \sigma, \tau = 1, \ldots, M),$$

where the left-hand side is the curvature tensor of W_M for the coordinate system $(\underset{0}{x}^1, \ldots, \underset{0}{x}^M)$. This result is known* and admits of a very simple proof by the use of Riemannian coordinates. Now if, at any point P of C, a two-space W_2 is constructed from all the geodesics of V_N tangential to V_2 at P, and if there is taken in W_2 a coordinate system $(\underset{0}{x}^1, \underset{0}{x}^2)$ such that at P

(7 . 3) $$\xi^i \equiv \frac{\partial x^i}{\partial u} = \frac{\partial x^i}{\partial \underset{0}{x}^1}, \quad \eta^i \equiv \frac{\partial x^i}{\partial v} = \frac{\partial x^i}{\partial \underset{0}{x}^2},$$

then†

(7 . 4) $$\underset{0}{G}_{1212} = G_{mnst}\, \xi^m \eta^n \xi^s \eta^t.$$

If K denotes the Gaussian curvature of W_2 at P (or, in other words, the Riemannian curvature of V_N for the directions ξ^i, η^i), then

(7 . 5) $$K\underset{0}{g} = \underset{0}{G}_{1212},$$

where

(7 . 6) $$\underset{0}{g} = \begin{vmatrix} \underset{0}{g}_{11} & \underset{0}{g}_{12} \\ \underset{0}{g}_{21} & \underset{0}{g}_{22} \end{vmatrix}$$

and

(7 . 7) $$\underset{0}{g}_{11}(dx^1)^2 + 2\underset{0}{g}_{12}(dx^1)(dx^2) + \underset{0}{g}_{22}(dx^2)^2$$

is the square of the line-element of W_2 at P‡. By virtue of (7 . 3) it is evident that

(7 . 8) $$\underset{0}{g}_{11}\, du^2 + 2\underset{0}{g}_{12}\, du\, dv + \underset{0}{g}_{22}\, dv^2$$

* Schouten, *loc. cit.*, 198.
† Cf. Bianchi, *loc. cit.*, 430.
‡ *Ibid.*, 428.

is the square of the line-element at V_2 at P. Therefore we may write

$$(7.9) \qquad I_2 = \int_{u_1}^{u_2} [\bar{\eta}^2 \sin^2 \theta(\boldsymbol{\xi}, \bar{\eta}) - K \underset{0}{g}]\, du,$$

where K is the Riemannian curvature of V_N for the directions ξ^i, η^i and $\underset{0}{g}$ is the determinant formed from the fundamental tensor of V_2 for the coordinate system (u, v).

8. *Second variation when the curve is geodesic and the end points are fixed.*

Turning to the second form for the second variation given in (6 . 18), we see that I_1 is zero if C is geodesic, since then $\boldsymbol{\xi}$ is zero. If the end points are fixed, η vanishes at $u = u_1$ and $u = u_2$ for all values of v under consideration, and therefore $\hat{\eta}$ vanishes at these points. Thus T is zero. Under these circumstances the second variation is

$$(8.1) \qquad \delta^2 L = \tfrac{1}{2}\delta v^2 \int_{u_1}^{u_2} [\bar{\eta}^2 \sin^2 \theta(\boldsymbol{\xi}, \bar{\eta}) - K \underset{0}{g}]\, du,$$

using the form of I_2 given in (7 . 9). Thus the second variation depends only on the direction and magnitude of the variation vector along C and in no other way on the nature of V_2.

We shall now assume that the variation vector is normal to C at every point, so that on C

$$(8.2) \qquad g_{ij}\xi^i \eta^j = 0.$$

This condition implies no restriction on the variations, the end points being fixed. It implies, however, a restricted choice of surface coordinates in V_2. Being given V_2 and an ∞^1 family of varied positions of C in V_2, defined as the parametric lines of u, it is only necessary to take new coordinates (u^*, v^*) such that the parametric lines of u^* coincide with the parametric lines of u and the parametric lines of v^* are the orthogonal trajectories of the parametric lines of u, together with the condition that $u^* = u$ on C. Then, dropping the asterisks, we know that (8 . 2) is true, and all results previously established are equally true for the new (u, v) coordinate system now adopted.

Differentiating (8 . 2) with respect to u, we obtain

$$g_{ij}\bar{\xi}^i\eta^j+g_{ij}\xi^i\bar{\eta}^j = 0, \tag{8.3}$$

and hence, since C is geodesic, we have on C

$$\bar{\eta}\cos\theta(\xi,\bar{\eta}) = 0. \tag{8.4}$$

Thus

$$\bar{\eta}^2\sin^2\theta(\xi,\bar{\eta}) = \bar{\eta}^2. \tag{8.5}$$

Furthermore, on C we have $\underset{0}{g_{11}}$ equal to unity, while $\underset{0}{g_{12}}$ vanishes. Thus we have

$$\underset{0}{g} = \underset{0}{g_{22}} = g_{ij}\eta^i\eta^j = \eta^2. \tag{8.6}$$

Substitution from (8 . 5) and (8 . 6) in (8 . 1) gives

$$\delta^2 L = \tfrac{1}{2}\delta v^2\int_{u_1}^{u_2}\left[\bar{\eta}^2-K\eta^2\right]du. \tag{8.7}$$

The following theorem results directly :

THEOREM XII.—*A curve C being geodesic, the second variation of its length is positive for all variations with fixed end points included in the type given in* § 1 (*and its length is therefore a relative minimum with respect to variations of that type*), *if the Riemannian curvature corresponding to every two-dimensional plane element containing the direction of C is zero or negative**. *The equivalent sufficient analytical condition is that $G_{ijkl}\xi^i\eta^j\xi^k\eta^l$ should be zero or negative at every point of C for arbitrary values of the components of η^m.*

9. *Expression for the second variation using the unit normal variation vector.*

Let us now further transform the expression for the second variation in the case where C is geodesic, the end points are fixed, and the v-curves cut C orthogonally, by the use of the *unit normal variation vector* μ^i defined by

$$\eta^i = \eta\mu^i. \tag{9.10}$$

* For the case $N = 2$, see Bolza, *loc. cit.*, 228.

We have then

(9.11) $$g_{ij}\mu^i\mu^j = 1,$$

and

(9.12) $$g_{ij}\xi^i\mu^j = 0.$$

Differentiation of these two equations with respect to u gives respectively

(9.13) $$g_{ij}\mu^i\bar{\mu}^j = 0,$$

and, since C is geodesic,

(9.14) $$g_{ij}\xi^i\bar{\mu}^j = 0.$$

Differentiation of (9.10) with respect to u gives

(9.15) $$\bar{\eta}^i = \dot{\eta}\mu^i + \eta\bar{\mu}^i,$$

and therefore, by (9.11) and (9.13),

(9.16) $$\bar{\eta}^2 = g_{ij}\bar{\eta}^i\bar{\eta}^j = \dot{\eta}^2 + \eta^2\bar{\mu}^2.$$

Introducing this expression into (8.7), we obtain the equation

(9.17) $$\delta^2 L = \tfrac{1}{2}\delta v^2 \int_{u_1}^{u_2} [\dot{\eta}^2 + (\bar{\mu}^2 - K)\eta^2]\, du.$$

An interesting geometrical result can easily be deduced from this equation. Let us at present consider only variations in some definite V_2, so that μ^i has definite values along C. We may treat V_2 as the fundamental manifold. Taking any coordinate system (x^1, x^2) in V_2, all our previous arguments apply, and (9.17) is the expression for the second variation with fixed end points, C being geodesic in V_2 if it is geodesic in V_N*.

In calculating (9.17) when V_2 is the fundamental manifold, η and $\dot{\eta}$ have the same values as in the case where V_N is fundamental; but $\bar{\mu}^i$ is the contravariant derivative calculated for the coordinate system in V_2, and K is the Gaussian curvature of V_2 itself (say K'). But, by (9.13) and (9.14), the vector $\bar{\mu}^i$ is perpendicular to two mutually perpendicular

* Cf. Bianchi, *loc. cit.*, 422.

vectors, namely μ^i and ξ^i. This is impossible in a space of two dimensions, and therefore $\bar{\mu}$ taken with respect to V_2 vanishes. Thus we have

$$\delta^2 L = \tfrac{1}{2}\delta v^2 \int_{u_1}^{u_2} [\dot{\eta}^2 - \Gamma\eta^2]\, du. \tag{9.18}$$

Comparison with (9 . 17) gives

$$\Gamma = K - \bar{\mu}^2, \tag{9.19}$$

where $\bar{\mu}$ is to be calculated with respect to V_N, and K is the Riemannian curvature of V_N corresponding to V_2. If $\bar{\mu}$ is zero,

$$\Gamma = K. \tag{9.20}$$

We may express these results in the following form :—

THEOREM XIII.—*At any point of a geodesic C of V_N the Gaussian curvature of any two-space V_2 containing C is equal to the excess of the Riemannian curvature of V_N corresponding to the element of V_2 over the square of the magnitude of the contravariant derivative with respect to C of the unit vector lying in V_2 and normal to C, calculated for the manifold V_N. The Gaussian curvature of V_2 is equal to the corresponding Riemannian curvature if, and only if, the unit normal vector is propagated parallelly along C.*

The following restricted theorem is of interest :—

THEOREM XIV.—*In a space V_N of constant and isotropic Riemannian curvature K there exists no two-space containing a geodesic of V_N and having a Gaussian curvature greater than K.*

We have already seen in Theorem XII that the second variation of the length-integral is positive if the Riemannian curvature corresponding to every element containing the direction of C is zero or negative. We shall now show that, if $N > 2$, it is always possible to construct a V_2 containing C, such that for variations in this V_2 the second variation is positive, on the sole assumption that K has a finite upper bound on C. The satisfaction of this condition is, in fact, a consequence of our postulates in § 1 concerning the fundamental tensor of V_N. From (9 . 17) we see that a sufficient condition for the second variation to be positive is

$$\bar{\mu}^2 \geqslant K \tag{9.21}$$

at all points of C. Let ϕ^i, ψ^i be two mutually perpendicular unit vectors, normal to C and propagated parallelly along C, and let α, β be two scalar functions of u. Write

$$\mu^i = \alpha\phi^i + \beta\psi^i. \tag{9.22}$$

Differentiation gives

$$\bar{\mu}^i = \dot{\alpha}\phi^i + \dot{\beta}\psi^i. \tag{9.23}$$

The vector μ^i defined by (9 . 22) is normal to C and is a unit vector if

$$\alpha^2 + \beta^2 = 1. \tag{9.24}$$

The value of $\bar{\mu}^2$ is

$$\bar{\mu}^2 = g_{ij}\bar{\mu}^i\bar{\mu}^j = \dot{\alpha}^2 + \dot{\beta}^2. \tag{9.25}$$

Now if we choose $\alpha = \cos Au$, $\beta = \sin Au$, where A is any constant greater than the upper bound of $\sqrt{K}$, (9 . 24) is satisfied, and (9 . 25) gives

$$\bar{\mu}^2 = A^2. \tag{9.26}$$

Thus (9 . 21) is satisfied, and we have the following result :—

THEOREM XV.—*Being given a geodesic C it is possible to find infinitely many two-spaces V_2 containing C, such that the second variation of the length of C is positive for all variations in V_2 between fixed end points.*

By (9 . 19) the equivalent geometrical statement is that it is always possible to draw infinitely many two-spaces through a geodesic having negative Gaussian curvature at every point of the geodesic. We can, in fact, draw such two-spaces with negative curvature exceeding in absolute value any preassigned positive number, however large.

Equation (9 . 17) expresses the second variation of the length-integral when C is geodesic and the curves of variation cut C orthogonally, the end points being fixed. In this equation $\eta\,|\delta v|$ is the magnitude of the infinitesimal variation vector, that is, the distance through which a point of C is displaced, and $\dot{\eta}\,|\delta v|$ is the rate of change of the magnitude of this displacement as we move along C. The magnitude of the displacement is arbitrary, except for the condition that it should vanish at the end points. If we take two two-spaces, V_2 and V_2^*, through C, we obtain an expression (9.17) for each of them. If η is the same function of u for V_2 and V_2^*, we shall say that the variations are *similar*. When this is the case, (9.17) enables us to compare the lengths of such varied curves adjacent to C lying in V_2 and V_2^* respectively. We may state the following theorem as a direct deduction from (9.17) :—

THEOREM XVI.—*If, at each point of a geodesic C, V_N has the same Riemannian curvature for all elements containing the direction of C, then, considering similar variations with fixed end points in all the two-spaces V_2 containing C, the second variation has a common value for all the V_2 formed by parallel propagation of a unit normal vector along C, and this common value is less than the second variation for any V_2 not generated in this manner.* Or, *for an infinitesimal normal variation of assigned magnitude at every point, the length of the varied curve is less when the unit normal vector is propagated parallelly than when it is not.*

10. *Conjugate points.*

From (9.17) we see that the Jacobian differential equation* for determination of the conjugate points for variations in any definite V_2 containing the geodesic C is

$$\frac{d^2y}{du^2}+(K-\bar{\mu}^2)\,y=0, \tag{10.1}$$

the conjugate points being consecutive zeros of any solution of this equation. Making use of Sturm's theorem†, the following theorem results :—

THEOREM XVII.—*If the Riemannian curvature corresponding to every element containing the direction of a geodesic C is less than a positive number A, the distance between a pair of conjugate points on C cannot be less than $\pi/\sqrt{A}$; while, if the Riemannian curvature is always greater than a positive number B, the distance between a pair of conjugate points for variations in a V_2 generated by parallel propagation of a unit normal vector is less than $\pi/\sqrt{B}$.*

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* Cf. Bolza, *loc. cit.*, 60.

† Darboux, *Leçons sur la théorie générale des surfaces*, 3 (1894), 100.